A MANUAL

OF

LOGARITHMIC COMPUTATION,

WITH NUMEROUS EXAMPLES.

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PREFACE.

THE objects of this Manual are—

First, to furnish to students full explanations of all the difficulties usually met with in the practice of computation by logarithms;

Secondly, to set before them compact and orderly forms of arrangement of the computations; and,

Thirdly, to provide instructors with an abundant collection of examples, progressively arranged, and covering all the points in regard to which mistakes are commonly made.

The Manual is the result of many years' experience of the difficulties met with in endeavoring to teach young students orderly and correct, as well as intelligent methods; and whatever may be thought of the success with which these difficulties have been treated, it is believed that not many of them have been overlooked.

The rather elaborate subdivision of some of the topics in the third chapter results from actual experience of the perplexity occasioned in the minds

of even pretty good students by the varying signs which characteristic and mantissa and their multipliers assume in different problems.

It is strongly recommended that the teacher insist upon the pupil's following, in all written exercises, the form given in the last solved example under each head. Orderly arrangement is almost indispensable to correct and rapid work, and quite indispensable to that thorough revision by the teacher, without which written exercises are, for most pupils, of little value.

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INTRODUCTION.

AN equation of the form $e^x = a$, in which the unknown quantity enters as an exponent, is called an exponential equation. The exponent x is called the logarithm of a in the system whose base is e. Accordingly,

The logarithm of any number is the exponent of the power to which a certain number, called the base of the system, must be raised, to produce the given number.

If $e^x = a$, and $e^{x'} = a'$, x and x' are the logarithms of a and a' in the system whose base is e.

Since $\frac{a}{a'} = \frac{e^x}{e^{x'}} = e^{x-x'}$, $x - x' = \log \frac{a}{a'}$, or $\log a - \log a' = \log \frac{a}{a'}$.

The logarithms of two numbers in the same system, therefore, are quantities whose difference is the logarithm of the ratio of the two numbers. The logarithms of numbers furnish, therefore, a means of finding, by subtraction alone, the ratios of the numbers, and were on that account called by Napier, their inventor, logarithms or ratio-numbers.

A table showing the exponents of the powers to which a given base must be raised, in order to produce successively all the natural numbers between certain limits, as 1 and 10000, is called a table of

logarithms. Any number may be taken as the base, and the values of the logarithms will depend on the value of the base. Thus if 2 be the base, we find, by raising 2 to the zero power, first power, and so on,

$2^0 = 1$,	therefore	$\log 1 = 0$	
$2^1 = 2$,	"	$\log 2 = 1$	
$2^2 = 4$,	"	$\log 4 = 2$	
$2^3 = 8$,	"	$\log 8 = 3$	
$2^4 = 16$,	"	$\log 16 = 4$	

The logarithm of 3 in this system is therefore between 1 and 2; the logarithms of 5, 6, and 7 are between 2 and 3, and so on.

The exact value of the logarithm of 3, in this system, is 1.584. As to the meaning of the statement that 2 must be raised to the 1.584th power to produce 3, this becomes plain when we recall that $e^{1.584} = e^{1+\frac{584}{1000}} = e^{\frac{1584}{1000}} = \sqrt[1000]{e^{1584}}$. If 4 be taken as the base, we have,

$4^0 = 1$,	therefore	$\log 1 = 0$,	as before
$4^1 = 4$,	"	$\log 4 = 1$,	instead of 2
$4^2 = 16$,	"	$\log 16 = 2$	
$4^3 = 64$,	"	$\log 64 = 3$	

In this system, therefore, the logarithms of 2 and 3 are between 0 and 1, the logarithms of 5, 6, 7, ... 15 are between 1 and 2, and so on.

Any number, as a, has therefore an infinite number of logarithms, and an infinite number of tables of logarithms is possible.

Of all these possible tables, only two have come into use. The first, called the natural, Naperian, or hyperbolic system, is the one first proposed by Napier. The second, called the common, Briggsian, or decimal system, is the one now generally used. The base of the latter is 10; that of the former is 2.7182818.........

The equation $e^x = a$ may be solved with respect to x in several ways.

1. The value of x may be developed into a continued fraction, as shown in elementary treatises on Algebra.* Giving to e any convenient value, and to a successive values differing by unity, between certain limits, the values of x may be found to any desired degree of approximation, and will be the logarithms of the successive values of a, in the system whose base is e. This method is so laborious as to be impracticable.

2. The equation $e^x = a$ may be transformed by the binomial theorem† into another, in which x shall appear as a factor rather than as an exponent. This equation is,

$$x\left[(e-1) - \tfrac{1}{2}(e-1)^2 + \tfrac{1}{3}(e-1)^3 \ldots\right] = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 \ldots,$$

$$\text{whence } x = \frac{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 \ldots}{(e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 \ldots}$$

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* Davies' Bourdon, Art. 248–255.

† Ibid., Art. 266.

which may be written,

$$x = M\left[(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 \ldots\right]$$

The value of x is thus found to be the product of two factors, one depending on e and the other on a, and each the sum of an infinite series. The sum of the first series, M, may be made to take any desired value, by assigning a proper value to e; and to Napier it seemed most convenient to make this sum equal to unity. The value of e, which gives this result, is 2.7182818...., and hence this number is the base of the Naperian system. In this system the logarithm of 1 is, as it is in every system, 0; the logarithm of 10 is 2 + a fraction, since the base 2.718.... must be raised to a power between the second and the third to produce 10; the logarithm of 100 is 4 + a fraction, and so on. There is here no easily observed or remembered relation between numbers and their logarithms.

The factor M, which Napier made unity, is called the modulus of the system.

Instead of assigning a simple value to the modulus of the system, and deducing the value of the base, Briggs assigned to the base of his system a simple value, 10, and deduced the corresponding modulus, 0.43429.... In the Briggsian system, which was immediately approved and adopted by Napier, the base being 10, the logarithm of 10 is 1, log $100 = 2$, log $1000 = 3$, etc.

Furthermore, the logarithm of a number between 10 and 100 must be between 1 and 2, and therefore

equal to 1 + a fraction; of a number between 100 and 1000, it is 2 + a fraction, and so on. These relations, being easily remembered, constitute one of the advantages of the Briggsian system.

3. The formula given above, $x = M\ [(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 \ldots]$ is not a convenient one for calculating the logarithm of a, even when M is assumed equal to unity, because a large number of terms in the series must be taken, in order to determine the value of x to a sufficient degree of approximation. This formula is therefore transformed into another.*

$$x = 2\ M \left[\frac{a-1}{a+1} + \frac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \ldots\right]$$

In this formula, since the numerator of each fraction is less than the denominator, each term is a proper fraction. The cube of such a fraction is less than the first power, the fifth power still less, and so the terms of the series diminish in value. Such a series is called a *converging* series, and an approximate value of the sum of such a series is found by taking a few terms. Thus, the logarithm of 3 in the Naperian system would be found from this formula to be

$$x = 2\ [\tfrac{1}{2} + \tfrac{1}{24} + \tfrac{1}{160} + \ldots]$$

from which the logarithm of 3 can be computed by using only a few of the terms of the series.

* Davies' Bourdon, Art. 270.

The utility of logarithms in calculations results from the following four principles:

1. Since $e^x = a$, or $x = \log a$,

and $e^{x'} = a'$, or $x' = \log a'$, we have

$$e^{x+x'} = aa', \text{ or } x + x' = \log aa';$$

that is, $\log aa' = \log a + \log a'$;

similarly, $\log aa'a'' \ldots . = \log a + \log a' + \log a'' + \ldots .$ or,

The logarithm of the product of two or more factors is equal to the sum of the logarithms of the factors.

2. From the same two equations

$$\text{we find } e^{x-x'} = \frac{a}{a'}, \text{ or } x - x' = \log \frac{a}{a'};$$

$$\text{that is, } \log \frac{a}{a'} = \log a - \log a', \text{or,}$$

The logarithm of a quotient is equal to the logarithm of the dividend, minus the logarithm of the divisor.

3. Again, since $e^x = a$, we find

$$e^{\frac{m}{n}x} = a^{\frac{m}{n}}; \text{ and if } n = 1,$$
$$e^{mx} = a^m, \text{ that is, } \log (a^m) = m \log a, \text{ or,}$$

The logarithm of the m^{th} power of a quantity is m times the logarithm of the quantity.

4. If in the equation $e^{\frac{m}{n}x} = a^{\frac{m}{n}}$, we make $m = 1$, we have $e^{\frac{x}{n}} = a^{\frac{1}{n}}$; that is, $\log (a^{\frac{1}{n}}) = \frac{x}{n}$, or,

The logarithm of the n^{th} root of a quantity is $\frac{1}{n}$th of the logarithm of the quantity.

The logarithm of a number consists, in general, of a whole number and a decimal fraction. The former is called the *characteristic*, and the latter the *mantissa*.*

The decimal part may be calculated to any number of places, depending on the degree of accuracy required. Four-place or even three-place tables are sufficient for rough approximation, while seven, ten, and even fourteen-place tables are used where great accuracy is required.†

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* *Mantissa*, something added, a make-weight.

† Loomis' six-place tables have been generally used in this work.

THE MANNER OF USING THE TABLE.

CHAPTER I.

TO FIND THE LOGARITHM OF A NUMBER.

I. When the number of significant figures is less than four.

1. The number being a whole number

a. Less than 100.

The logarithms of the first 100 numbers are usually printed entire, opposite the numbers themselves, and may therefore be taken directly from the table without any trouble. Thus we find

Log 11 = 1.041393.	Log 51 = 1.707570.
Log 25 = 1.397940.	Log 99 = 1.995635.
Log 60 = 1.778151.	Log 73 = 1.863323.

Ex. 1, log 45.	Ex. 3, log 77.	Ex. 5, log 91.
Ex. 2, log 33.	Ex. 4, log 85.	Ex. 6, log 96.

b. Between 100 and 1000.

In this part of the table the characteristic is not printed, its value being always less by unity than

the whole number of figures in the given number. To find the mantissa, look in the column of numbers for the first three figures of the given number, and at the top of the page for the fourth figure, if there is one. The last four figures of the logarithm are found at the intersection of the row and column thus determined, and the first two at the beginning of the row in the column headed 0. Thus we find,

$$\text{Log } 1140 = 3.056905.$$
$$\text{Log } 2343 = 3.369772.$$
$$\text{Log } 447 = 2.650308.$$

Ex. 7, log 4205.	Ex. 12, log 799.	Ex. 17, log 5799.
Ex. 8, log 7988.	Ex. 13, log 487.	Ex 18, log 6933.
Ex. 9, log 727.	Ex. 14, log 8099.	Ex. 19, log 800.
Ex. 10, log 5444.	Ex. 15, log 1066.	Ex. 20, log 7082.
Ex. 11, log 9569.	Ex. 16, log 191.	Ex. 21, log 5525.

In the column headed 0, all the six figures of the mantissa are printed, but as the first two figures continue the same in a number of logarithms in succession, they are not reprinted in the following columns. When, however, the increase of the last four figures amounts to so much as to cause an increase of one of the two leading figures, it is necessary either to print the two leading figures in the column in which the change takes place, or to call attention to the change. The latter is done by printing dots instead of the zeros, of which one or more must in such case follow the leading figures Thus we find,

Log 2343 = 3.369772.

Log 2344 = 3.369958.

Log 2345 = 3.370143, which last logarithm is printed .143, indicating that the third figure is 0, and that the first two figures are to be taken from the line below, opposite 235. Even when the last four figures of the logarithm do not contain any of these dots, it is of course still necessary to take the two leading figures from the line below, if the change has taken place before in the same line.

Of course other figures besides the third may be zeros, as in the following cases:—

Log 3312 = 3.520090, printed . . 90.
Log 6607 = 3.820004, printed . . . 4.

Ex. 22, log 2633.	Ex. 27, log 2048.	Ex. 32, log 8129.
Ex. 23, log 7764.	Ex. 28, log 5627.	Ex. 33, log 6166.
Ex. 24, log 8914.	Ex. 29, log 3236.	Ex. 34, log 2637.
Ex. 25, log 1869.	Ex. 30, log 3098.	Ex. 35, log 1178.
Ex. 26, log 2138.	Ex. 31, log 5755.	Ex. 36, log 1125.

c. Beyond 10000, but not containing more than four significant figures.

The equation logarithm 1146 = 3.059185 means $10^{3.059185} = 1146$. If we multiply both numbers by 10, we have, $10^{4.059185} = 11460$, or log 11460 = 4.059185. Similarly $10^{5.059185} = 114600$, or log 114600 = 5.059185, and so on. So that

The logarithms of all whole numbers which consist of the same significant figures, arranged in the same order, have the same mantissa, and the characteristic of each

is less by unity than the number of integer figures in the given number. Thus

$$\text{Log } 21380 = 4.330008.$$
$$\text{Log } 446700 = 5.650016.$$

Ex. 37, log 501900.	Ex. 42, log 200000.	Ex. 47, log 204200.
Ex. 38, log 57590.	Ex. 43, log 316300.	Ex. 48, log 3313000.
Ex. 39, log 56240.	Ex. 44, log 97770000.	Ex. 49, log 100000.
Ex. 40, log 4570.	Ex. 45, log 46780.	Ex. 50, log 323000.
Ex. 41, log 977400.	Ex. 46, log 10990.	Ex. 51, log 33990.

2. The number being a mixed number.
From the equation

Log 1146 = 3.059185, or $10^{3.059185} = 1146$, we find, by dividing by 10,

$$10^{2.059185} = 114.6, \text{ or } \log 114.6 = 2.059185.$$
Similarly, $10^{1.059185} = 11.46$, or $\log 11.46 = 1.059185$.

From these examples it appears that

The mantissa of the logarithm of a mixed number is found by treating the number as a whole number; and the characteristic is less by unity than the number of integer figures in the given number. Thus,

$$\text{Log } 14.4 = 1.158362.$$
$$\text{Log } 154.9 = 2.190051.$$
$$\text{Log } 1.446 = 0.160168.$$

Ex. 52, log 5.361.	Ex. 57, log 99.99.	Ex. 62, log 7.944.
Ex. 53, log 8.081.	Ex. 58, log 1.549.	Ex. 63, log 7.245.
Ex. 54, log 9.009.	Ex. 59, log 9.999.	Ex. 64, log 97.73.
Ex. 55, log 6.121.	Ex. 60, log 218.8.	Ex. 65, log 478.9.
Ex. 56, log 93.33.	Ex. 61, log 57.55.	Ex. 66, log 549.6.

3. The number being a decimal.

Returning to the example

$$10^{1.059185} = 11.46, \text{ or } \log 11.46 = 1.059185,$$

we find, by dividing repeatedly by 10,

$$10^{0.059185} = 1.146, \text{ or } \log 1.146 = 0.059185$$

$$10^{0.059185-1} = 0.1146, \text{ or } \log 0.1146 = 0.059185 - 1$$
$$= -0.940815$$

$$10^{0.059185-2} = 0.01146, \text{ or } \log 0.01146 = 0.059185 - 2$$
$$= -1.0940815$$

and so on. It thus appears that *the logarithms of decimal fractions are negative*, as they should be, since, to produce a fraction less than unity by means of the powers of 10, unity must be divided by some power of 10, that is, 10 must be raised to some power indicated by a negative exponent. The fraction 0.01, for instance, or $\frac{1}{100}$, is, $\frac{1}{10^2}$; that is $10^{-2} = 0.01$ or $\log 0.01 = -2$.

From the examples it appears that the logarithm of a decimal fraction and the logarithm of a whole number composed of the same figures, not only have different signs, but have different numerical values. If the tables, therefore, are to give the true values of the logarithms directly, two different tables will be required: one giving the logarithms of whole numbers, and the other the logarithms of decimal fractions. That this may be avoided, and the tables greatly simplified will be evident if we note that in the last example given,

$$10^{0.059185-1} = 0.1146, \text{ or } \log 0.1146 = 0.059185 - 1,$$

if we keep the logarithm in this form, or in the

equivalent form $-1+0.059185$, the decimal part of the logarithm of 0.1146 is the same as that of the logarithm of 11.46, 114.6, etc., and may therefore be taken directly from the same table. The characteristic, however, is -1, which is written $\bar{1}$, to indicate that the negative sign belongs, not to the entire logarithm, but only to the characteristic, the complete logarithm being $-1 + 0.059185$, which is written $\bar{1}.059185$.

In the same manner we find

$$\text{Log } 0.01146 = \bar{2}.059185.$$

$$\text{Log } 0.00001146 = \bar{5}.059185, \text{ and so on.}$$

Whence,

The mantissa of the logarithm of a decimal fraction is found by treating the number as a whole number; and the characteristic is negative and greater by unity than the number of zeros which follow the decimal point.

Ex. 67, log 0.144.
Ex. 68, log 0.0149.
Ex. 69, log 0.0000801.
Ex. 70, log 0.008081.
Ex. 71, log 0.0000999.
Ex. 72, log 0.1651.
Ex. 73, log 0.009009.
Ex. 74, log 0.0000077.
Ex. 75, log 0.00000002.
Ex. 76, log 0.0000102.
Ex. 77, log 0.0001025.
Ex. 78, log 0.0176.
Ex. 79, log 0.7555.
Ex. 80, log 0.1009.
Ex. 81, log 0.0001009.

II. When the number of significant figures is greater than four. Interpolation.

1. Whole number.

The logarithm of such a number as 11403 is not given directly in a table which extends only to

10000. It is evident, however, that the characteristic of the logarithm will be 4, and that the mantissa will be the same as that of the logarithm of 1140.3, and will lie between the mantissas corresponding to 1140 and 1141. The finding of the value of a quantity which lies thus between two quantities in a table is called *interpolation.*

If the logarithms increased uniformly from log 1140 to log 1141, the decimal part of log 1140.3 would exceed that of 1140 by $\frac{3}{10}$ of the difference between log 1140 and log 1141. Though the logarithms do not increase uniformly, they do so very nearly, and the error committed in supposing the increase to be uniform is in general very small. When the utmost accuracy is required, the methods of interpolation given in treatises on algebra must be used.

We find in the table,

$$\begin{aligned} \text{Log } 1141 &= 3.057286 \\ \text{Log } 1140 &= 3.056905 \end{aligned}$$

Difference 381

$\frac{3}{10}$ of the difference = 114.3

This added to 3.056905

$$\begin{aligned} \text{gives, log } 1140.3 &= 3.057019 \\ \text{or, log } 11403 &= 4.057019 \end{aligned}$$

The figure 381, which is the difference between the two successive logarithms between which the required logarithm lies, is called the *tabular* difference, and is always printed in the tables, in a column

headed D, or Diff. The value of any number of tenths of this difference which it may be necessary to add to the logarithm taken from the table, is easily computed, after a little practice, without writing, and should be so computed whenever it is possible. Thus, in the example above, it is 3×38.1 or 114.3. The third figure of the tabular difference must not be neglected, for, when multiplied by the given number of tenths, the product may exceed five, and in this case the last figure must be increased by unity. Thus, if the tabular difference were 382, we should have $3 \times 38.2 = 114.6$, which is nearer 115 than 114. When accuracy rather than speed is required, this point is an important one.

When the number whose logarithm is required contains more than five figures, it is necessary to add, not only tenths but hundredths of the tabular difference.

Thus, $\log 577933 = \log 577900 + \frac{33}{100}$ Diff.; but in this case also the interpolation is easily effected, for since

$$\begin{aligned} D &= 75, \text{ we find} \\ \tfrac{3}{10} D &= 22.5 \\ \tfrac{3}{100} D &= 2.25 \\ \text{and therefore } \tfrac{33}{100} D &= 24.75, \text{ or } 25. \\ \text{Hence, } \log 577933 &= 5.761853 + 0.000025 \\ &= 5.761878. \end{aligned}$$

The operation of interpolation is simplified by using tables like those of Loomis, which give, generally under the head "Proportional Parts," the values of $\frac{1}{10}$, $\frac{2}{10}$, etc., of the tabular difference. A tenth of each of these quantities then gives the value of $\frac{1}{100}$, $\frac{2}{100}$, etc. Thousandths, ten-thousandths, etc.,

are found in the same way. Thus, to find log 20813, we first find,

Log 20810	4.318272
The tabular difference being 208, we find at the bottom of the page (Loomis' Logs), $\frac{3}{10}$ 208 =	62
Therefore, log 20813	4.318334

To find log 3100759, we have

Log 3100000		6.491362
D = 140 ; therefore,	0.7 D	98
	0.05 D	7
	0.009 D	1.3
Hence, log 3100759		6.491468

The calculation may be arranged in the same form when the proportional parts are not given, the multiplications being performed mentally. Thus, to find Log 18642168.

Log 18640000	7.270446
	46
	2.3
	1.4
	.2
	7.270496 *

* It is recommended that computations be arranged in the form given in the last example solved under each head. The teacher should insist upon this, in order to secure neat and orderly work, and such as can be easily revised.

Ex. 82, log 911002.	Ex. 87, log 20420171.	Ex. 92, log 2909099.
Ex. 83, log 7006001.	Ex. 88, log 48980777.	Ex. 93, log 100188.
Ex. 84, log 6762005.	Ex. 89, log 812939.	Ex. 94, log 18509.
Ex. 85, log 7763829.	Ex. 90, log 4668779.	Ex. 95, log 192401.
Ex. 86, log 7246001.	Ex. 91, log 67619.	Ex. 96, log 827279.

In a six-place table of logarithms extending from 100 to 10000, the tabular difference varies from 0.000434 to 0.000043. The fractional parts of this difference which are required for interpolation vary therefore as follows:

$\frac{1}{10}$ D,	from 43.4	millionths	to 4.3	millionths.
$\frac{1}{100}$ D,	" 4.34	"	" 0.43	"
$\frac{1}{1000}$ D,	" 0.434	"	" 0.043	"

In the first part of the table, therefore, where D is 434 millionths, the correction of the logarithm for a fifth figure in the number will be from 1 to 9 times 43.4 millionths, for a sixth figure it will be a multiple of 4.34, for a seventh, of 0.434, and for an eighth, of 0.0434. Now, the largest multiple of this last figure, namely, 9×0.0434 would be only 0.3906, and would therefore not affect the sixth figure of the logarithm. It is, therefore, impossible, even in this part of the table, to express exactly, by means of six-place logarithms, the logarithms of numbers which contain more than seven significant figures. Thus, in the two examples on page 16, the corrections for the seventh figure are respectively 1.3 and 1.4; and the corrections for the eighth figure add nothing to the sixth figure of the logarithm.

In the last part of the table, the tabular difference

being only one-tenth as great as in the first part, not more than six significant figures in the number will have any effect on the logarithm.

It thus appears that it is useless, in using six-place tables, to carry the interpolation by proportional parts beyond thousandths of the tabular difference.

2. Mixed numbers and decimal fractions.

It is evident, from what has been said before, that the logarithm of a mixed number or a decimal fraction differs from that of the corresponding whole number only in the characteristic.

Thus we find

Log 44.068.......................1.644122.
Log 5.77932......................0.761878.
Log 0.0056254....................$\bar{3}$.750154.

The following examples are numbers that occur frequently in calculation, and the logarithms of which, therefore, are very useful:

Ex. 97, log	3.1415926.	(Ratio of circumference of circle to diameter.)
Ex. 98, log	365.24224.	(Tropical year in mean solar days.)
Ex. 99, log	39.139.	(Length of seconds pendulum in inches.)
Ex. 100, log	32.1808.	(Acceleration due to gravity in latitude 45°.)
Ex. 101, log	3.2809.	(Meter in feet.)
Ex. 102, log	39.3708.	(Meter in inches.)
Ex. 103, log	0.6214.	(Kilometer in miles.)
Ex. 104, log	0.30479.	(Foot in meters.)
Ex. 105, log	2.5399.	(Inch in centimeters.)

Ex. 106, log	1.6093.	(Mile in kilometers.)
Ex. 107, log	2.471.	(Hectar in acres.)
Ex. 108, log	119.6	(Ar in square yards.)
Ex. 109, log	0.4543.	(Acre in hectars.)
Ex. 110, log	35.3166.	(Ster in cubic feet.)
Ex. 111, log	0.908.	(Liter in quarts, dry measure.
Ex. 112, log	1.0567.	(Liter in U. S. quarts, liquid measure.)
Ex. 113, log	0.8804.	(Liter in imperial quarts.)
Ex. 114, log	0.7645.	(Cubic yard in sters.)
Ex. 115, log	1.1013.	(Quart, dry, in liters.)
Ex. 116, log	1.1359.	(Imperial quart in liters.)
Ex. 117, log	1.1544.	(Beer quart " ")
Ex. 118, log	35.24	(U. S. bushel " ")
Ex. 119, log	36.323.	(Imperial bushel in liters.)
Ex. 120, log	3.6244.	(Cord in sters.)
Ex. 121, log	15.4323.	(Gram in grains Troy.)
Ex. 122, log	0.0353.	(Gram in ounces Avoirdupois.)
Ex. 123, log	2.2046.	(Kilogram in pounds Avoirdupois.)
Ex. 124, log	0.45359.	(Pound Avoirdupois in kilograms.)
Ex. 125, log	0.37324.	(Pound Troy or Apothecary's in kilos.)
Ex. 126, log	0.19295.	(Franc in dollars.)
Ex. 127, log	5.1826.	(Dollar in francs.)

CHAPTER II.

TO FIND THE NUMBER CORRESPONDING TO A GIVEN LOGARITHM.

THIS number is sometimes called the *anti-logarithm* of the given logarithm.

I. When the exact logarithm is contained in the table.

The logarithm being found in the table, the first three figures of the number will be found to the left of the logarithm, in the column of numbers, and the fourth figure at the top of the column. The characteristic of the logarithm being less by unity than the number of integer figures in the number, if the characteristic is less than three, one or more of the last figures belong to the decimal part of the required number. If the characteristic is greater than three, one or more zeros must be annexed to the four figures found, to make the number of integer figures greater by unity than the characteristic. Thus, we find,

Number whose logarithm is 1.863323 = 73.00.

This is written, Log^{-1} 1.863323, and read as above.

Similarly, Log^{-1} 0.290035 = 1.95.

Log^{-1} 4.470116 = 29520.00.

Ex. 128, $\log^{-1}$ 1.863323.	Ex. 133, $\log^{-1}$ 3.347135.
Ex. 129, $\log^{-1}$ 0.290035.	Ex. 134, $\log^{-1}$ 4.892484.
Ex. 130, $\log^{-1}$ 3 891705.	Ex. 135, $\log^{-1}$ 7.862489.
Ex. 131, $\log^{-1}$ 5.028978.	Ex. 136, $\log^{-1}$ 4.397766.
Ex. 132, $\log^{-1}$ 2.222196.	Ex. 137, $\log^{-1}$ 3.242044.

The only point requiring especial attention in this case is that whenever the last four figures of any logarithm or of one preceding it in the same horizontal line contain one of the dots which stand for zeros, the first two figures are to be found in the line below. Thus, if it is required to find

Log^{-1} 3.270213, we find the first two figures, 27, and do not find 0213 opposite them; but above, in the line 26, we find .213. The figures 27, though found in the line below, belong therefore with these, making the given logarithm .270213, and the corresponding number 1863 is found to the left *of the last four figures.* Similarly, we find

$$\text{Log}^{-1}\ \bar{1}.500099 = 0.3163.$$
$$\text{Log}^{-1}\ \bar{3}.640084 = 0.004366.$$

Ex. 138, $\log^{-1}$ 1.944483.	Ex. 143, $\log^{-1}$ 5.518382.
Ex. 139, $\log^{-1}$ 1.000434.	Ex. 144, $\log^{-1}$ $\bar{1}$.650016.
Ex. 140, $\log^{-1}$ 0.000434.	Ex. 145, $\log^{-1}$ $\bar{2}$.700011.
Ex. 141, $\log^{-1}$ $\bar{3}$.000434.	Ex. 146, $\log^{-1}$ $\bar{3}$.300161.
Ex. 142, $\log^{-1}$ 4.340047.	Ex. 147, $\log^{-1}$ $\bar{2}$.900039.

II. When the given logarithm is not found in the table. Interpolation.

If the given logarithm falls between two logarithms in the table, the required number falls between the two corresponding numbers, and the process of find-

ing this number is again *interpolation.* The inverse process, like the direct process described on page 13, is based on the assumption that the differences of the numbers and the differences of the logarithms are proportional to each other.

Let it be required to find the $\log^{-1} 2.220250$.
The nearest logarithm in the table

is $\log 166.0 = 220108$

The difference is	142
The tabular difference is	261

The number required, therefore, lies between 166.0 and 166.1, and exceeds the former by $\frac{142}{261}$ of the difference between the two. Converting $\frac{142}{261}$ into a decimal fraction, we find 0.54. The difference between the required number and 166.0 is, therefore, 0.54×0.1 and the required number is 166.054.

Required, $\log^{-1} 4.320182$

$\log^{-1} 4.320146 = 20900.0$

D = 207) 36.00 (1.7
207
1530
1449
Ans.—20901.7

Required, $\log^{-1} 1.330098$

$\log^{-1} 1.330008 = 21.38$

D = 203) 90.00 (44
Ans.—21.3844

Required, $\log^{-1} \bar{2}.340281$

$\log^{-1} \bar{2}\,340246 = 0.02189$

$$
\begin{array}{rl}
D = & 199)\,35.00\,(\quad 17 \\
& \underline{199} \\
& 1510 \\
& 1393
\end{array}
$$

Ans.—0.0218917

The reduction of the vulgar fraction to a decimal, in order to find the figures that may be required beyond the four given in the table, is facilitated, just as the direct interpolation is, by a table of proportional parts. Thus, in the first of the examples given above,

Log^{-1} 2.220250, we find

Log^{-1} 2.220108 = 166.0

Diff	142	
Tabular Diff	261	
In the table of proportional parts, we find that the nearest difference to 142 is 131, which is $\frac{5}{10}$ of 262, and we therefore add $\frac{5}{10}$ of 0.1....................	=	.05
But 142 exceeds 131 by 11, which we find in the table of proportional parts, is $\frac{4}{100}$ of 261, and we therefore add $\frac{4}{100}$ of 0.1		.004
Thus finding the required number,		166.054

The reduction of the vulgar fraction to a decimal may be carried to any desired number of places; but a limit is soon reached, beyond which the further extension of the reduction will be useless.

We have seen (p. 17) that a table of six-place logarithms gives only the logarithms of numbers which do not contain more than six or seven figures. When the calculation has been carried to the seventh significant figure therefore, nothing is gained by going farther. Thus, in the last example, after adding $\frac{4}{100}$ of 0.1, getting

	166.054
we might have proceeded as follows:	
$11 - 10.5 = 0.5$. The next number below this in the table of proportional parts is $0.26 = \frac{1}{1000}$ D. We therefore add $\frac{1}{1000}$ of $0.1 =$	.0001
Again, $0.5 - 0.26 = 0.24$. The next number below this is $0.21 = \frac{8}{10000}$ D. We therefore add $\frac{8}{10000}$ of $0.1 =$	.00008
and so on. Hence the required logarithm is	166.05418

But all the figures after the seventh (or after the sixth in this part of the table) being uncertain, we write

$$\text{Log}^{-1}\, 2.220250 = 166.054$$

To find the $\log^{-1}$ 4.523346, we find in the table,

	$\log^{-1}$ 4.523226 =	33360.0
Diff.	120	
Nearest diff. in prop. parts,	117 = $\frac{9}{10}$ D ∴	9.0
	3 = $\frac{4}{100}$ D ∴	.4
Hence,		*Ans.*—33369.4

Similarly we find $\log^{-1}$ 8.760190 = 57569130.

Ex. 148,	$\log^{-1}$	5.734521,	to one	decimal	place.
Ex. 149,	"	5.622406,	"	"	"
Ex. 150,	"	4.253459,	"	"	"
Ex. 151,	"	4.132181,	"	"	"
Ex. 152,	"	5.481151,	"	"	"
Ex. 153,	"	4.510029,	"	"	"
Ex. 154,	"	2.630080,	"	"	"
Ex. 155,	"	1.650058,	3	"	places.
Ex. 156,	"	0.660320,	"	"	"
Ex. 157,	"	0.510109,	"	"	"
Ex. 158,	"	$\bar{2}$.570193,	6	"	"
Ex. 159,	"	$\bar{1}$.021389,	"	"	"
Ex. 160,	"	$\bar{1}$.000689,	"	"	"
Ex. 161,	"	$\bar{1}$.000264,	"	"	"
Ex. 162,	"	$\bar{2}$.908156,	"	"	"
Ex. 163,	"	$\bar{1}$.900080,	"	"	"
Ex. 164,	"	$\bar{3}$.750560,	"	"	"
Ex. 165,	"	$\bar{2}$.690911,	"	"	"
Ex. 166,	"	$\bar{1}$.492300,	"	"	"
Ex. 167,	"	$\bar{1}$.500622,	"	"	"
Ex. 168,	"	$\bar{2}$.030555,	"	"	"

CHAPTER III.

ARITHMETICAL OPERATIONS BY MEANS OF LOGARITHMS.

I. Multiplication.

1. Both factors positive.

a. Both entire.

To find the product of 187 and 4291.

$$\begin{aligned} \text{From the table, } \log\ 187 &= 2.271842 \\ \log 4291 &= \underline{3.632559} \end{aligned}$$

By Principle 1, Introduction,

$$\log \text{product} = 5.904401$$

$$\text{Hence, Product} = \quad 802419.$$

By actual multiplication, however, we find the product to be 802417. The error of two units results from the fact that the logarithms of the two factors are only approximately correct. Using seven-place logarithms, we find, log 187 . . . . 2.2718416

$$\begin{aligned} \log 4291 \ldots\ldots & \underline{3.6325585} \\ & 5.9044001 \end{aligned}$$

$$\text{Product} = \quad 802417$$

The source of the error will now be easily understood, and the degree of accuracy attainable with

a given table can be readily estimated. It appears that the six-place logarithms of 187 and 4291 are too large respectively by 4 and 5 ten-millionths, nearly (not exactly, because even seven-place logarithms are only approximate), and their sum, therefore, is too large by 9 ten-millionths. But, the tabular difference being 540 ten-millionths, the sum of the errors is $\frac{9}{540}$, or nearly 0.02 of this, and the number 802419 is too great by 0.02 of the difference between 802400 and 802500, that is, by 2 units.

It will sometimes happen that the two errors in the logarithms will be of opposite signs; and in such a case, if they are also equal, the sum of the two logarithms will be the correct logarithm of the required product. Thus, let it be required to find the product of 195 and 1743. We find

log 195....	2.290035
log 1743....	3.241297
	5.531332

Product....339885.

In this case, the seven-place logarithms of the factors being 2.2900346 and 3.2412974, it is easily seen that the two errors in the six-place logarithms balance each other.

In six-place tables, as ordinarily printed, it is impossible to tell whether the last figure is too large or too small, and whether, therefore, the total error in a calculated product is due to the sum or the difference of two errors.

To estimate the degree of accuracy attainable with a given table, we observe that, as any one of the logarithms in a six-place table may be in error to the extent of 0.000000499......, or say to any amount less than half a unit in the sixth place, the sum of two such logarithms may be in error to any amount between 0 and $\pm \frac{1}{1000000}$; that is to say, the error is numerically less than one unit in the sixth place. But it has been seen (page 17), that the tabular difference varies from 0.000434 to 0.000043, and that one unit in the sixth place corresponds to anything between $\frac{1}{431}$ and $\frac{1}{43}$ of a unit in the fourth place of the antilogarithm. As $\frac{1}{431} = 0.0023$, and $\frac{1}{43} = 0.023$, an error of a unit in the sixth place of a logarithm corresponds to an error of not less than 0.0023 of a unit, and not more than 0.023 of a unit in the fourth figure of the antilogarithm, or not less than two units in the seventh figure, nor more than two in the sixth. The product of two numbers, therefore, as determined by six-place logarithms may be in error to an amount varying from 0 to plus or minus two units in the sixth figure, according to the signs and values of the errors of the logarithms of the factors, and the part of the table in which the logarithm of the product is found.

Seven-place tables usually give the logarithms of all numbers up to 100000 (that is, of five figures) without interpolation.* In such tables the logarithm is uncertain to the extent of 0.00000049......, and

* See Vega's Logarithmic Tables.

therefore the error in the sum of two logarithms may have any value numerically less than one unit in the seventh place. But, the tabular difference in such tables varying from 0.0000434 to 0.0000043, an error of a unit in the seventh place corresponds to an error of not less than 0.0023 nor more than 0.023 of a unit in the fifth place, that is, of not less than two units in the eight place, nor more than two in the seventh. The product of two numbers, therefore, as calculated by means of seven-place logarithms, is subject to an error varying from 0 to $\pm$ two units in the seventh place.

The following examples will illustrate the several cases that may arise.

Both errors negative: product too great.

Ex. 169,	2596×296.	Ex. 171,	3995×239.
Ex. 170,	2896×335.	Ex. 172,	5861×254.

Both errors positive: product too small.

Ex. 173,	1848×395.	Ex. 175,	5844×948.
Ex. 174,	7083×788.	Ex. 176,	4588×968.

Errors contrary and equal: product correct.

Ex. 177,	1174×272.	Ex. 178,	1591×122.

In all these examples seven-place logarithms would give the correct product. If the number of significant figures in the product is seven or more, the result obtained by seven-place logarithms is uncertain. When great accuracy is required, larger tables must be used in such cases.

b. One factor or both mixed.

Product of 794.416 and 1.391.

$$\begin{array}{lrl} \text{Log } 794.416 &=& 2.900048 \\ \text{Log } \quad 1.391 &=& \underline{0.143327} \\ && 3.043375 \\ && \quad\ \ \underline{362} \\ && \qquad 13 \end{array}$$

Product = 1105.03.

Ex. 179,	1921.0	×	7.423.
Ex. 180,	7764.25	×	7.76.
Ex. 181,	4898.03	×	5.496.
Ex. 182,	1600.0	×	0.0257.
Ex. 183,	1.012	×	1.055.
Ex. 184,	3.1416	×	16.421.
Ex. 185,	3000.1	×	10.39 to seven figures.
Ex. 186,	3821.17	×	100.16 " "
Ex. 187,	4976.23	×	1177.3 " "
Ex. 188,	1.012	×	1.055 " "

c. One factor or both decimal.

Product of 0.4951 and 0.3875.

$$\begin{array}{lrl} \text{Log } \ 0.4951 &=& \bar{1}.694693 \\ \text{Log } \ 0.3875 &=& \underline{\bar{1}.588272} \\ \text{Log product} &=& \bar{1}.282965 \end{array}$$

Product = 0.1918.

The sum of the characteristics above is $\bar{2}$. Carrying 1 from the sum of the mantissas makes $\bar{2}+1$ or $\bar{1}$.

Ex. 31.10 × 0.114.

$$\begin{aligned} \text{Log } 31.10 &= 1.492760 \\ \text{Log } 0.114 &= \bar{1}.056905 \\ \hline &\ 0.549665 \end{aligned}$$

Product = 3.545.

Ex. 1002 × 0.000066.

$$\begin{aligned} \text{Log } 1002.0 &= 3.000868 \\ \text{Log } 0.000066 &= \bar{5}.819544 \\ \hline &\ \bar{2}.820412 \end{aligned}$$

Product = 0.06613.

Ex. 0.00712 × 0.000025.

$$\begin{aligned} \text{Log } 0.00712 &= \bar{3}.852480 \\ \text{Log } 0.000025 &= \bar{5}.397940 \\ \hline &\ \bar{7}.250420 \end{aligned}$$

Product = 0.000000178.

Ex. 189,	331.4	×	0.02455.
Ex. 190,	56245	×	0.00079.
Ex. 191,	1.421	×	0.2183.
Ex. 192,	0.00741	×	0.006166.
Ex. 193,	0.101	×	0.0202.
Ex. 194,	87.97	×	0.0292.

Ex. 195,	1116.11	×	0.000513.
Ex. 196,	0.0707	×	0.0091.
Ex. 197,	0.0209	×	0.0595.
Ex. 198,	0.799	×	0.899.

2. Two factors, one or both negative, whole, mixed or decimal.

The logarithms of negative quantities are all imaginary, since the base of the system is positive, and no power of a positive quantity can be negative. Nevertheless, since the *numerical* value of the product of two or more factors is the same, whatever be the signs of the factors, the product may be calculated as if the factors were both positive, and then the proper sign given to the result. Thus,

Product of 6.761 and − 123.75.

$$\begin{aligned} \text{Log } 6.767 &= 0.830396 \\ \text{Log } 123.75 &= 2.092545n \\ &\overline{2.922941n} \end{aligned}$$

Product = − 837.4.

The n's after the second and third logarithms indicate that the corresponding *numbers*, not the logarithms, are negative.

Product of − 0.01233 and − 0.00061.

$$\begin{aligned} \text{Log } 0.01233 &= \bar{2}.090963n \\ \text{Log } 0.00061 &= \bar{4}.785330n \\ &\overline{\bar{6}.876293} \end{aligned}$$

Product, 0.000007521.

In this example, both factors being negative, the product is positive.

Ex. 199,	117.57	×	− 0.0404.
Ex. 200,	− 1.325	×	− 0.176.
Ex. 201,	0.00007587	×	− 489.9.
Ex. 202,	− 0.00133	×	0.2231.
Ex. 203,	1.792	×	− 0.866.
Ex. 204,	− 20.002	×	− 11.021.
Ex. 205,	− 1.74	×	− 3.74.
Ex. 206,	− 0.023	×	− 5.022.
Ex. 207,	− 0.013	×	− 0.066.

3. More than two factors.

Product of 171.2, − 411.4, − 31.07, 0.021, − 0.003.

$$\begin{aligned}
\text{Log } 171.2 &= 2.233504 \\
\text{Log } 411.4 &= 2.614264n \\
\text{Log } 31.07 &= 1.492341n \\
\text{Log } 0.021 &= \bar{2}.322219 \\
\text{Log } 0.003 &= \bar{3}.477121n \\
&\overline{\ 2.139449n}
\end{aligned}$$

Product, − 137.8.

The number of negative factors being odd, the product is negative.

Ex. 208, (− 51.004) × (− 3.071) × (− 0.000202).
Ex. 209, (− 617.21) × 4.0012 × 118.22 × 3.51.
Ex. 210, (− 0.033) × 0.079 × (− 0.421).
Ex. 211, (− 17.0026) × (− 9.211) × (− 0.007).
Ex. 212, (− 0.025) × 0.0211 × 11.71 × (− 0.29).

The considerations set forth on pages 27–29, will enable the computer to estimate the uncertainty in any of the above products.

II. Division.

1. Terms both positive; whole, decimal, or mixed.

a. Characteristics of both logarithms positive (that is, both numbers greater than unity).

(1.) Both characteristic and mantissa of the logarithm of the dividend greater than those of the divisor (that is, more integer figures in the dividend than in the divisor, and the figures of the dividend larger than those of the divisor):

Ex. $\frac{182.4}{12.18}$.

$$\begin{aligned} \text{Log } 182.4 &= 2.261025 \\ \text{Log } \ \ 12.18 &= \underline{1.085647} \end{aligned}$$

By Principle 2, Introduction,
we have, log quotient $= 1.175378$

Hence, quotient $= 14.97$.

In subtracting logarithms, particularly when there are several subtractions, time is saved by using the *Arithmetical Complement* or *Cologarithm,* which is *the remainder obtained by subtracting the logarithm from* 10. If we *add* the arithmetical complement (written A. C. or Colog.) instead of subtracting the loga-

rithm itself, the result is obviously too great by 10. We correct it, therefore, by subtracting 10 from the result. Thus, to divide 179.3 by 10.4, we have,

$$\begin{aligned}
\text{Log } 179.3 &= 2.253580 \\
10 - \log 10.4 = 10 - 1.017033 &= \underline{8.982967} \\
\text{Hence, } 10 + \log \text{quotient} &= 11.236547 \\
\text{Log quotient} &= 1.236547
\end{aligned}$$

$$\text{Quotient} = 17.24.$$

It is not necessary to perform in writing the operation of subtracting the logarithm from 10, but the remainder may be taken directly from the table. Thus, in the example above, since, in subtracting 1.017033 from 10.000000, we subtract the last significant figure from 10, and then have to carry one, or increase each of the remaining figures by 1, we may begin at the left, and, subtracting each figure from 9, except the last *significant* figure, subtract that from 10. The result, 8.982967, may be written, after a very little practice, as readily as the logarithm 1.017033 itself. The operation of subtracting 10 from the sum of course needs no writing.

Ex. $\dfrac{712.5}{63.2}$.

$$\begin{aligned}
\text{Log } 712.5 &= 2.852785 \\
\text{A. C. log } 63.2 &= \underline{8.199283} \\
& 1.052068
\end{aligned}$$

$$\text{Quotient} = 11.27.$$

Examples to be solved by means of the arithmetical complement or cologarithm.

Ex. 213, $\frac{5731}{128}$. Ex. 218, $\frac{62.71}{3.99}$.

Ex. 214, $\frac{7994.6}{100.32}$. Ex. 219, $\frac{194.2}{1.68}$.

Ex. 215, $\frac{49.72}{3.216}$. Ex. 220, $\frac{73.3}{3.32}$.

Ex. 216, $\frac{41165.2}{18.3}$. Ex. 221, $\frac{187.5}{162.1}$.

Ex. 217, $\frac{11701.01}{1.09}$. Ex. 222, $\frac{19.02}{1.82}$.

(2.) Characteristic of the logarithm of the dividend greater than that of the divisor, and mantissa less (that is, number of integers greater, but values of the integers less in the dividend than in the divisor).

Ex. $\frac{753.3}{89.42}$.

$$\begin{array}{lcr} \text{Log } 753.3 & = & 2.876968 \\ \text{Log } 89.42 & = & 1.951435 \\ \hline & & 0.925533 \end{array}$$

Quotient, 8.424.

In this case, when the difference of the mantissas is taken, there is 1 to be carried; but the difference of the characteristics will still be positive or 0, inasmuch as the characteristic of the minuend is the greater.

Performing the same operation by means of the cologarithm, we have

$$\begin{array}{lcl} \text{Log } 753.3 & = & 2.876968 \\ \text{Colog } 89.42 & = & \underline{8.048565} \\ & & 0.925533 \end{array}$$

as before.

Examples to be solved by means of the arithmetical complement.

Ex. 223, $\frac{1701.4}{293.7}$.

Ex. 224, $\frac{44.22}{8.82}$.

Ex. 225, $\frac{916.3}{99.0}$.

Ex. 226, $\frac{711.0}{8.1}$.

Ex. 227, $\frac{119.3}{20.04}$.

Ex. 228, $\frac{501.18}{69.2}$.

Ex. 229, $\frac{86.401}{9.7}$.

Ex. 230, $\frac{119.3}{1.81}$.

Ex. 231, $\frac{7429.1}{9.31}$.

Ex. 232, $\frac{11.66}{9.42}$.

(3.) Characteristic and mantissa of the logarithm of the dividend less than those of the divisor (that is, number and value of the integers in the dividend less than in the divisor).

Ex. $\frac{59.63}{6821.0}$.

Log	59.63	1.775465
Log	6821.0	3.833848
		$\bar{3}.941617$

Quotient, 0.008742.

Or, by means of the cologarithm,

Log	59.63	1.775465
Colog	6821.0	6.166152
		$\bar{3}.941617$

as before.

Examples by means of the cologarithm.

Ex. 233, $\dfrac{7.916}{891.12}$. Ex. 238, $\dfrac{55.91}{612.3}$.

Ex. 234, $\dfrac{516.02}{7921.0}$. Ex. 239, $\dfrac{891.17}{9241.1}$.

Ex. 235, $\dfrac{75.06}{891.7}$. Ex. 240, $\dfrac{6.002}{99.003}$.

Ex. 236, $\dfrac{1.003}{12.17}$. Ex. 241, $\dfrac{11.7}{812.4}$.

Ex. 237, $\dfrac{79.29}{801.3}$. Ex. 242, $\dfrac{51.3}{888.8}$.

b. Either characteristic or both negative (that is, either number or both less than unity).

(1.) Mantissa of the logarithm of the dividend greater than that of the divisor. Either characteristic greater.

Ex. $\dfrac{295.0}{0.187}$.

Log 295.0	2.469822
Log 0.187	$\bar{1}.271842$
	3.197980

Quotient, 1577.54.

The difference of the mantissas, which are both positive, is 0.197980. The difference of the characteristics is $2 - (-1)$, or 3.

Using the arithmetical complement, we have,

Log 295	2.469822
Colog 0.187	10.728158
	3.197980

Ex. $\dfrac{20.62}{0.0012}$.

Log 20.62	1.314289
Colog 0.0012	12.920819
	4.235108

Quotient, 17183.

Ex. $\dfrac{0.00912}{0.0722}$.

Log 0.00912	$\bar{3}.959995$
A. C. log 0.0722	11.141463
	$\bar{1}.101458$

Quotient, 0.1263.

Examples by means of cologarithm.

Ex. 243, $\frac{7.49}{0.06912}$.

Ex. 244, $\frac{0.62916}{22.15}$.

Ex. 245, $\frac{0.179}{0.174}$.

Ex. 246, $\frac{0.03001}{0.02002}$.

Ex. 247, $\frac{0.9127}{0.00084}$.

Ex. 248, $\frac{0.06013}{0.012}$.

Ex. 249, $\frac{0.00029}{0.00013}$.

Ex. 250, $\frac{0.0002}{194.0001}$.

Ex. 251, $\frac{8.321}{0.132}$.

Ex. 252, $\frac{0.5902}{6.387}$.

(2.) Mantissa of the logarithm of the dividend less than that of the divisor.

In this case, on arriving at the characteristic in the subtraction, there will be 1 to carry, which, therefore, will be subtracted from the difference of the characteristics when that difference is positive, and added when negative. Thus,

Ex. $\frac{0.00749}{0.08912}$.

Log 0.00749	$\bar{3}.874482$
Log 0.08912	$\bar{2}.949975$
	$\bar{2}.924507$

Here, the difference of the mantissas is $\bar{1}.924507$, and the difference of the characteristics is $\bar{1}$. Hence the logarithm of the quotient is $\bar{2}.924507$.

Or, by means of the cologarithm,

Log	0.00749	$\bar{3}.874482$
A. C. log	0.08912	11.050025
		$\bar{2}.924507$

Quotient, 0.08404.

Ex. 253, $\frac{0.08201}{0.0962}$. Ex. 258, $\frac{11.402}{0.502}$.

Ex. 254, $\frac{0.00023}{0.05111}$. Ex. 259, $\frac{0.0000101}{0.0000204}$.

Ex. 255, $\frac{8.000055}{0.9722}$. Ex. 260, $\frac{0.392}{0.914}$.

Ex. 256, $\frac{0.3920}{6.4011}$. Ex. 261, $\frac{0.00101}{0.00201}$.

Ex. 257, $\frac{0.71}{901.93}$. Ex. 262, $\frac{6.527}{0.699}$.

2. Either term or both negative, whole, decimal, or mixed; either term greater.

The sign of the quotient, like that of the product in multiplication, is positive if the terms have the same sign, negative if they have contrary signs. The operations are identical with those of case 1 and its subdivisions.

Ex. $\frac{0.00291}{-16.0385}$.

Log	0.00291	$\bar{3}.463893$
A. C. log	16.0385	$8.794834n$
		$\bar{4}.258727n$

Quotient, – 0.0001814.

Ex. 263, $\dfrac{-90.211}{0.752}$.

Ex. 264, $\dfrac{1.002}{-3.721}$.

Ex. 265, $\dfrac{-0.0401}{-3.872}$.

Ex. 266, $\dfrac{-3.009}{2.75}$.

Ex. 267, $\dfrac{-601.1}{0.02}$.

Ex. 268, $\dfrac{105.38}{-2.71}$.

Ex. 269, $\dfrac{-3.387}{-6.666}$.

Ex. 270, $\dfrac{-0.3535}{17.4}$.

Ex. 271, $\dfrac{16.201}{-12.001}$.

Ex. 272, $\dfrac{-2.821}{0.003}$.

The error in division, due to the inexactness of the logarithms, is the *algebraic* difference (which is the *arithmetical* sum or difference) of the two errors. The uncertainty is therefore the same as that in the product of two numbers.

III. Multiplication and division; factors whole, decimal, or mixed, positive or negative.

The rules for this case are obvious, after simple multiplication and division are understood. The only point of importance to be noticed is the saving effected by the use of the arithmetical complement.

Ex. $\dfrac{11.2 \times 86400 \times (-3.2)}{-18.1 \times (-0.0021)}$.

Log 11.2		1.049218
Log 86400		4.936514
Log 3.2		0.505150n
		6.490882n
Log 18.1	1.257679n	
Log 0.0021	$\bar{3}$.322219n	
		$\bar{2}$.579898
		7.910984n

Hence we have, carrying the interpolation to seven figures,

Value of the fraction, $-$ 81477540.

Using arithmetical complements, we find

Log 11.2	1.049218
Log 86400	4.936514
Log 3.2	0.505150n
A. C. log 18.1	8.742321n
A. C. log 0.0021	12.677781n
	7.910984n,

rejecting 20 from the sum because two arithmetical complements have been used.

It will be observed that the second calculation is shorter than the first, by two lines of figures, and two operations of addition and subtraction, not counting the operation of rejecting 20 and the two subtrac-

tions performed in finding the arithmetical complements, none of which require any writing.

Ex. 273, $\frac{31.23 \times 173.02}{0.21 \times (-1.43)}$. Ex. 275, $\frac{716.1 \times (0.024)}{(-0.091)(-0.002)}$.

Ex. 274, $\frac{62.3 \times (-0.124)}{(-3.291)(-3.74)}$. Ex. 276, $\frac{11.17 \times (-3.002)}{(-0.006)(-0.8412)(-16.24)}$.

IV. Involution.

Involution and evolution may be considered as the same process, the exponent being in the first case always a whole number, and in the second always a proper fraction. Cases also occur in which the exponent is a mixed number. These, as shown on page 3, may be considered as involving both operations.

1. Number positive.
   a. Number whole or mixed.
      (1.) Exponent positive.

Ex. 289^2.

By Principle 3, Introduction,

$$\text{Log } 289^2 = 2 \log 289.$$

Log 289	2.460898
	2
Log 289^2	4.921796

$$289^2 = 83521.$$

Ex. 8318.52^2.

Log 8318.52	3.920046
	2
	7.840092

Hence, $8318.52^2 = 69,197,900$, to within 100, the interpolation having been carried to six figures.

Ex. 109.2^{12}.

Log 109.2	2.038223
	12
	24.458676

$109.2^{12} = 28,752,530\ldots\ldots$ to 25 figures, of which only the first seven are determined by six-place logarithms.

Ex. 277,	311.07^5.	Ex. 282,	1.75^8.
Ex. 278,	5.029^{10}.	Ex. 283,	20.009^5.
Ex. 279,	1802.0^4.	Ex. 284,	21.17^6.
Ex. 280,	270.4^3.	Ex. 285,	3.008^5.
Ex. 281,	1.003^5.	Ex. 286,	1511.02^3.

In involution, the error in the logarithm of the number being multiplied by the index of the power, the result will be inaccurate when the multiplier is large. The principles set forth on page 27, will enable the student to determine when more accurate tables must be used.

(2.) Exponent negative.

Ex. 218.8^{-2}.

$$
\begin{array}{lr}
\text{Log } 218.8 & 2.340047 \\
 & -2 \\
\hline
 & -4.680094 \\
 & =\bar{5}+1-0.680094 \\
 & =\bar{5}+0.319906. \\
 & =\bar{5}.319906.
\end{array}
$$

Hence, $218.8^{-2} = 0.00002089.$

The same result is reached more simply as follows:

$$218.8^{-2} = \frac{1}{218.8^2}. \quad \text{Hence,}$$

$$
\begin{array}{lr}
\text{Log } \quad 1 & 0.000000 \\
2 \log 218.8 & 4.680094 \\
\hline
\text{Log } 218.8^{-2} & \bar{5}.319006
\end{array}
$$

It is evidently not necessary to write log 1. The remainder, $\bar{5}.319906$, is the arithmetical complement of 2 log 218.8, minus 10.

Ex. 316.3^{-3}.

$$
\begin{array}{lr}
\text{Log } 316.3 & 2.500099 \\
 & 3 \\
\hline
 & 7.500297 \\
\text{A. C.} - 10 & \bar{8}.499703
\end{array}
$$

$316.3^{-3} = 0.0000000316.$

Ex. 287,	7.3001^{-4}.	Ex. 292,	11.75^{-2}.
Ex. 288,	20.0201^{-2}.	Ex. 293,	11.9116^{-3}.
Ex. 289,	1.09^{-1}.	Ex. 294,	1.1^{-1}.
Ex. 290,	16.202^{-1}.	Ex. 295,	20.02^{-3}.
Ex. 291,	182.47^{-1}.	Ex. 296,	17.42^{-3}.

b. Number a fraction.

(1.) Exponent positive.

Ex. 0.2295^3.

$$\begin{array}{ll} \text{Log } 0.2295 & \bar{1}.360783 \\ & \underline{3} \\ & \bar{2}.082349 \end{array}$$

$$0.2295^3 = 0.01208.$$

The product of the characteristic and the exponent is negative (in this case -3). From this we have to subtract whatever is carried from the mantissa, which in this case is 1.

Ex. 297,	0.005^3.	Ex. 302,	0.2043^4.
Ex. 298,	0.916^2.	Ex. 303,	0.1124^6.
Ex. 299,	0.3^5.	Ex. 304,	0.0186^3.
Ex. 300,	0.84135^3.	Ex. 305,	0.7215^2.
Ex. 301,	0.1073^5.	Ex. 306,	0.3999^3.

(2.) Exponent negative.

Ex. 0.0139^{-3}.

$$
\begin{array}{lr}
\text{Log } 0.0139 & \bar{2}.143015 \\
 & -3 \\
\hline
 & +6 - 0.429045 \\
 & = 5 + 0.570955 \\
 & = \quad 5.570955.
\end{array}
$$

Or, since $0.0139^{-3} = \dfrac{1}{0.0139^3}$,

$$
\begin{array}{lr}
\text{Log } 0.0139 & \bar{2}.143015 \\
 & 3 \\
\hline
 & \bar{6}.429045 \\
\text{A. C.} - 10 & 5.570955,
\end{array}
$$

as on page 46. Hence, $0.0139^{-3} = 372{,}353$.

Ex. 0.000357^{-4}.

$$
\begin{array}{llr}
\text{Log } 0.000357 & & \bar{4}.552668 \\
 & & 4 \\
\hline
 & & \overline{16} + 2.210672 \\
 & = & \overline{14}.210672 \\
\text{A. C.} - 10 & = & 13.789328
\end{array}
$$

Hence, $0.000357^{-4} = 6156 \ldots\ldots$ to 14 figures.

Ex. 307, 0.0139^{-3}.
Ex. 308, 0.7944^{-3}.
Ex. 309, 0.2692^{-4}.
Ex. 310, 0.0309^{-2}.
Ex. 311, 0.9334^{-1}.
Ex. 312, 0.003236^{-1}.
Ex. 313, 0.007245^{-5}.
Ex. 314, 0.6951^{-3}.
Ex. 315, 0.60002^{-1}.
Ex. 316, 0.0717^{-2}.

2. Number negative.

The numerical value of the result is found as in the preceding case. The sign of the result is positive for even powers and negative for odd ones.

a. Number whole or mixed.

(1.) Exponent positive.

Ex. $(-8.319)^4$.

Log 8.319	$0.920071n$
	4
	$3.680284n$

$$(-8.319)^4 = 4789.43.$$

Ex. $(-7.021)^5$.

Log 7.021	$0.846399n$
	5
	$4.231995n$

$$(-7.021)^5 = -17060.6.$$

Ex. 317, $(-221.2)^2$.
Ex. 318, $(-1.008)^3$.
Ex. 319, $(-8122)^2$.
Ex. 320, $(-7.0591)^3$.
Ex. 321, $(-11.04)^3$.
Ex. 322, $(-824)^4$.
Ex. 323, $(-5.91)^4$.
Ex. 324, $(-1.091)^5$.
Ex. 325, $(-21.881)^2$.
Ex. 326, $(-537.35)^2$.

(2.) Exponent negative.

Since $a^{-n} = \frac{1}{a^n}$, and since the denominator a^n will be positive for even and negative for odd powers, the sign of the result is determined as in the preceding case.

Ex. 327, $(-1709)^{-3}$.
Ex. 328, $(-20.21)^{-4}$.
Ex. 329, $(-4.1173)^{-5}$.
Ex. 330, $(-9.777)^{-2}$.
Ex. 331, $(-47.87)^{-2}$.
Ex. 332, $(-2.089)^{-3}$.
Ex. 333, $(-18.21)^{-4}$.
Ex. 334, $(-2.0041)^{-7}$.
Ex. 335, $(-199.6)^{-3}$.
Ex. 336, $(-77.642)^{-3}$.
Ex. 337, $(-0.9998)^{-2}$.

b. Number a decimal fraction, exponent positive or negative.

Ex. $(-0.0124)^4$.

Log 0.0124	$\bar{2}.093422n$
	4
	$\bar{8}.373688$

$$(-0.0124)^4 = 0.00000002364.$$

Ex. $(-0.083)^5$.

Log 0.083	$\bar{2}.919078n$
	5
	$\bar{6}.595390n$

$$(-0.083)^5 = -0.000003939.$$

Ex. $(-0.0102)^{-3}$.

Log 0.0102	$\bar{2}.008600n$
	3
	$\bar{6}.025800n$
A. C. − 10 (see page 46)	5.974200

$$(-0.0102)^{-3} = -942325.$$

Ex. 338, $(-\ 0.000204)^{-2}$.

Ex. 339, $(-\ 0.181)^{3}$.

Ex. 340, $(-\ 0.3972)^{3}$.

Ex. 341, $(-\ 0.919)^{-3}$.

Ex. 342, $(-\ 0.181)^{5}$.

Ex. 343, $(-\ 0.3163)^{4}$.

Ex. 344, $(-\ 0.421)^{-3}$.

Ex. 345, $(-\ 0.7763)^{-2}$.

Ex. 346, $(-\ 0.32)^{-4}$.

Ex. 347, $(-\ 0.871)^{-2}$.

V. Evolution.

This operation is the same as that of involution, except that the exponent is a proper fraction.

1. Number positive.

   a. Whole or mixed number.

   (1.) Exponent positive.

Ex. $\sqrt[2]{41.7} = 41.7^{\frac{1}{2}}$.

Log 41.7	1.620136
	$\frac{1}{2}$
	0.810068

$$\sqrt{41.7} = 6.457.$$

Ex. 348, $101.25^{\frac{1}{5}}$.

Ex. 349, $211.171^{\frac{1}{9}}$.

Ex. 350, $106.021^{\frac{1}{6}}$.

Ex. 351, $624.3^{\frac{1}{4}}$.

Ex. 352, $989201^{\frac{1}{6}}$.

Ex. 353, $19082^{\frac{1}{6}}$.

Ex. 354, $54.97^{\frac{1}{3}}$.

Ex. 355, $776.31^{\frac{1}{7}}$.

Ex. 356, $1.738^{\frac{1}{3}}$.

Ex. 357, $2.138^{\frac{1}{7}}$.

(2.) Exponent negative.

Ex. $73.2^{-\frac{1}{3}}$.

Log 73.2	1.864511
	$-\frac{1}{3}$
	-0.621504
Adding and subtracting unity, we have	$\bar{1}+1-0.621504$
	$=\bar{1}.378496$
or, as on page 46,	
$\frac{1}{3}$ log 73.2	0.621504
A. C. $-$ 10	$\bar{1}.378496$

Hence, $73.2^{-\frac{1}{3}} = 0.23905$.

Ex. $79{,}965{,}000^{-\frac{1}{3}}$.

Log 79,965,000	7.902900
	$\frac{1}{3}$
	2.634300
A. C. $-$ 10	$\bar{3}.365700$

0.0023211.—*Ans.*

Ex. 358, $1.216^{-\frac{1}{7}}$.

Ex. 359, $36.19^{-\frac{1}{4}}$.

Ex. 360, $112.02^{-\frac{1}{4}}$.

Ex. 361, $384.2^{-\frac{1}{6}}$.

Ex. 362, $811.16^{-\frac{1}{3}}$.

Ex. 363, $74.23^{-\frac{1}{5}}$.

Ex. 364, $4467.0^{-\frac{1}{2}}$.

Ex. 365, $726.9^{-\frac{1}{3}}$.

Ex. 366, $2.952^{-\frac{1}{4}}$.

Ex. 367, $75.88^{-\frac{1}{3}}$.

b. Number a fraction.

(1.) Exponent positive.

Ex. $0.2042^{\frac{1}{2}}$.

Log 0.2042	$\bar{1}.310056 = \bar{1} + 0.310056$
Dividing the characteristic by 2 we have	$-\ 0.500000$
" " mantissa " "	$+\ 0.155028$
and adding, "	$-\ 0.344972$
which is the same as	$\bar{1} + 1 - 0.344972$
or	$\bar{1}.655028$

0.4519.—*Ans.*

The same result would have been reached, and more simply, if we had, before dividing by the exponent of the root, *added to the characteristic* -1, *or such other number as would make it exactly divisible by the index, and subtracted the same number from the mantissa.* Thus,

Log 0.2042	$\bar{1}.310056$
Adding and subtracting 1,	$\bar{2} + 1.310056$
Dividing by the index,	$\bar{1} + 0.655028$
	$= \bar{1}.655028$

as before.

Ex. $\sqrt[5]{0.00003233}$.

$$\text{Log } 0.00003233 \qquad \bar{5}.590606$$
$$\frac{1}{5}$$
$$\bar{1}.101921$$

0.1264.—*Ans.*

Ex. $\sqrt[4]{0.000005241}$.

$$\begin{array}{lr} \text{Log } 0.000005241 & \bar{6}.719414 \\ & = \bar{8} + 2.719414 \\ & \underline{\tfrac{1}{4}} \\ & \bar{2}.679853 \end{array}$$

0.04784.—*Ans.*

Ex. 368, $0.001986^{\frac{1}{3}}$.

Ex. 369, $0.002999^{\frac{1}{4}}$.

Ex. 370, $0.89965^{\frac{1}{6}}$.

Ex. 371, $0.00005555^{\frac{1}{2}}$.

Ex. 372, $0.564112^{\frac{1}{5}}$.

Ex. 373, $0.786541^{\frac{1}{3}}$.

Ex. 374, $0.0022951^{\frac{1}{4}}$.

Ex. 375, $0.010101^{\frac{1}{3}}$.

Ex. 376, $0.21881^{\frac{1}{4}}$.

Ex. 377, $0.004784^{\frac{1}{5}}$.

(2.) Exponent negative.

In this case as in the preceding one, it is most convenient to add and subtract such number as will make the characteristic exactly divisible by the index of the root.

Ex. $0.000204^{-\frac{1}{3}}$.

$$\begin{array}{lr} \text{Log } 0.000204 & \bar{4}.309630 \\ \text{Adding and subtracting 2,} & \bar{6} + 2.309630 \\ & \underline{\tfrac{1}{3}} \\ & \bar{2}.769877 \\ \text{A. C.} - 10 \text{ (page 46.)} & 1.230123 \end{array}$$

16.99.—*Ans.*

Ex. 378, $0.02138^{-\frac{1}{4}}$.

Ex. 379, $0.5496^{-\frac{1}{5}}$.

Ex. 380, $0.1712^{-\frac{1}{3}}$.

Ex. 381, $0.7991^{-\frac{1}{6}}$.

Ex. 382, $0.08129^{-\frac{1}{8}}$.

Ex. 383, $0.0009773^{-\frac{1}{4}}$.

Ex. 384, $0.0001866^{-\frac{1}{5}}$.

Ex. 385, $0.00141^{-\frac{1}{6}}$.

Ex. 386, $0.013191^{-\frac{1}{6}}$.

Ex. 387, $0.162244^{-\frac{1}{9}}$.

2. Number negative, whole, mixed or fractional, exponent positive or negative.

The method of proceeding in this case is the same as when the number is positive; only it must be observed that all even roots are imaginary.

Ex. $\sqrt[3]{-28.7}$.

Log 28.7	1.457882n
	$\frac{1}{3}$
	0.485961n

$-$ 3.061.—*Ans.*

Ex. $\sqrt[4]{-71.2}$ imaginary.

Ex. $(-592.1)^{-\frac{1}{3}}$. $-$ 0.1190.—*Ans.*

Ex. $(-1622)^{\frac{1}{9}}$. $-$ 2.2734.—*Ans.*

Ex. 388, $(-2900.02)^{-\frac{1}{5}}$. Ex. 393, $(-72.45)^{-\frac{1}{5}}$.

Ex. 389, $(-1.206)^{-\frac{1}{3}}$. Ex. 394, $(-0.7763)^{-\frac{1}{3}}$.

Ex. 390, $(-1600.0)^{\frac{1}{4}}$. Ex. 395, $(-0.0211)^{-\frac{1}{4}}$.

Ex. 391, $(-22.92)^{\frac{1}{3}}$. Ex. 396, $(-0.8172)^{\frac{1}{2}}$.

Ex. 392, $(-8.129)^{-\frac{1}{5}}$. Ex. 397, $(-0.399)^{-\frac{1}{3}}$.

VI. Involution and evolution.

When the exponent of the power to which a quantity is to be raised is a fraction of which both numerator and denominator are whole numbers greater than unity, or either of them a mixed number or a decimal, the operation indicated involves both involution and evolution. Such expressions are

$$a^{\frac{3}{2}} = \sqrt[2]{a^3}.$$

$$a^{\frac{129}{171}} = \sqrt[171]{a^{129}}.$$

$$a^{3.71} = a^{\frac{371}{100}} = \sqrt[100]{a^{371}}.$$

$$a^{0.56} = a^{\frac{56}{100}} = \sqrt[100]{a^{56}}.$$

$$a^{\frac{1}{1.19}} = a^{\frac{1}{\frac{119}{100}}} = a^{\frac{100}{119}} = \sqrt[119]{a^{100}}.$$

$$a^{\frac{1}{0.39}} = a^{\frac{1}{\frac{39}{100}}} = a^{\frac{100}{39}} = \sqrt[39]{a^{100}}.$$

Such expressions occur frequently in the applications of mathematics to physics and engineering. The values of such expressions can ordinarily not be found by means of arithmetic alone, but they are easily found by means of logarithms, in the same way as those of the similar expressions already discussed in which the index of the power is a whole number or a fraction.

I. Number positive.

1. Exponent positive.

a. Number whole or mixed.

(1.) Numerator and denominator of the exponent both whole numbers.

Ex. $56^{\frac{7}{8}}$.

$$\begin{array}{lr} \text{Log } 56 & 1.748188 \\ & 7 \\ \hline & 8)\ 12.237316 \\ \hline & 1.529664 \end{array}$$

$$56^{\frac{7}{8}} = 33.86.$$

The multiplication of the logarithm of the number by the exponent of the power may be performed by logarithms, as explained on page 26; but, when the terms of the exponent are small, actual multiplication will be easier. Solving Ex. 398 throughout by logarithms, we should have,

Log 6327....3.801198.

$$
\begin{array}{ll}
\text{Log log } 6327 \ldots\ldots\ldots\ldots & 0.579920 \\
\text{Log } 221 \ldots\ldots\ldots\ldots\ldots\ldots & 2.344392 \\
\text{Colog } 119 \ldots\ldots\ldots\ldots\ldots & 7.924453 \\
\hline
\text{Log } (\tfrac{221}{119} \text{ log } 6327) \ldots\ldots\ldots & 0.848765
\end{array}
$$

$$\tfrac{221}{119} \text{ log } 6327 = \text{log } 6327^{\frac{221}{119}} \ldots\ldots 7.059365$$

$$\text{Hence, } 6327^{\frac{221}{119}} = 11469500.$$

For the reasons explained on pages 21 and 32, the result is only approximate.

Ex. 398, $6327^{\frac{221}{119}}$.	Ex. 403, $1.239^{\frac{3}{2}}$.
Ex. 399, $77.63^{\frac{3}{11}}$.	Ex. 404, $1.015^{\frac{5}{8}}$.
Ex. 400, $4.898^{\frac{17}{16}}$.	Ex. 405, $2.959^{\frac{3}{7}}$.
Ex. 401, $30.09^{\frac{3}{7}}$.	Ex. 406, $2.638^{\frac{7}{5}}$.
Ex. 402, $72.45^{\frac{11}{5}}$.	Ex. 407, $269.9^{\frac{7}{6}}$.

(2.) Numerator of the exponent a mixed number or a decimal.

Ex. $12^{2.37} = 12^{\frac{237}{100}}$.

$$
\begin{array}{lr}
\text{Log } 12 & 1.079181 \\
 & 2.37 \\
\hline
 & 7554267 \\
 & 3237543 \\
 & 2158362 \\
\hline
 & 2.557659
\end{array}
$$

$$12^{2.37} = 361.13$$

Such expressions may be multiplied or divided, the one by the other, by the methods already explained. Thus,

Ex. $\frac{3.1^{3.55}}{2.3^{1.7}}$.

$$\text{Log}\,\frac{3.1^{3.55}}{2.3^{1.7}} = 3.55 \log 3.1 + \text{A. C. } 1.7 \log 2.3$$

Log 3.1	0.491362	Log 2.3	0.361728
	3.55		1.7
	2456810		2532096
	2456810		361728
	1474086		0.614938
	1.744335	A. C.	9.385062
			1.744335
			1.129397

13.47.—*Ans.*

Performing the multiplications by logarithms, the computation will be arranged as follows:

Log 3.1	0.491362.		
	Log 0.491362	$\bar{1}.691402$	
	Log 3.55	0.550228	
	Log (3.55 × log 3.1)	0.241630	
	3.55 × log 3.1		1.744341

Log 2.3	0.361728.		
Log 0.361728		$\bar{1}.558383$	
Log 1.7		0.230449	
Log (1.7 × log 2.3)		$\bar{1}.788832$	
1.7 × log 2.3			0.614940
Log quotient			1.129401
Quotient 13.47.			

The following is the solution of Ex. 408 by seven place logarithms.

21.1....1.3242825....0.1219808			
	3.3	0.5188139	
		0.6404947	
Log $21.1^{3.3}$..............			4.3701333
15.0....1.1760913....0.0704410			
	4.27	0.6304279	
		0.7008689	
Log $15.0^{4.27}$.............			5.0219093
Log product,			9.3920426
Product.......2466281200.			

Ex. 408, $21.1^{3.3} \times 15^{4.27}$. Ex. 413, $14.31^{7.3}$.

Ex. 409, $10.16^{3.81}$. Ex. 414, $1129^{1.21}$.

Ex. 410, 1.24^{278}. Ex. 415, $20.02^{3.3} \times 1.06^{2.2}$.

Ex. 411, $\dfrac{3.18^{2.51}}{17.1^{3.6}}$. Ex. 416, $\dfrac{4.19^{1.7}}{3.21^{2.5}}$.

Ex. 412, $103.3^{11.2}$. Ex. 417, $4.899^{5.5}$.

(3.) Denominator or both terms of the exponent mixed numbers or decimal.

Ex. $3.17^{\frac{1}{0.54}} = 3.17^{\frac{100}{54}} = \sqrt[54]{3.17^{100}}$.

$$
\begin{array}{ll}
\text{Log } 3.17 & 0.501059 \\
 & 100 \\
\hline
 & 54)50.105900(0.927887 \\
 & 48\,6 \\
 & 150 \\
 & 108 \\
 & 425 \\
 & 378 \\
 & 479 \\
 & 432 \\
 & 470 \\
 & 432 \\
 & 380
\end{array}
$$

$3.17^{\frac{1}{0.54}} = 8.470.$

Ex. 418, $93.33^{\frac{1}{0.15}}$.

Ex. 419, $8.319^{\frac{2}{0.41}}$.

Ex. 420, $1236^{\frac{5}{1.55}}$.

Ex. 421, $1.779^{\frac{3}{2.56}}$.

Ex. 422, $14.49^{\frac{5}{0.91}}$.

Ex. 423, $199.9^{\frac{7}{0.16}}$.

Ex. 424, $1.629^{\frac{4}{1.13}}$.

Ex. 425, $20.49^{\frac{3}{1.16}}$.

Ex. 426, $1.418^{\frac{4}{0.91}}$.

Ex. 427, $3.17^{\frac{5}{12.5}}$.

b. Number a decimal fraction.

In this case, the characteristic of the logarithm being negative and the mantissa positive, attention must be paid to the signs in multiplying. It will be well to indicate the negative part of each partial product by a line drawn over it, as in the case of negative characteristics. In adding the partial products, each figure must then be taken with its proper sign.

(1.) Numerator and denominator of the exponent both whole.

Ex. $0.0211^{\frac{5}{7}}$.

$$\begin{array}{lr} \text{Log } 0.0211 & \overline{2}.324282 \\ & 5 \\ \hline & 7)\overline{9}.621410 \\ & \overline{2}.803059 \end{array}$$

Adding and subtracting -5, before dividing by 7 (see page 53), we have $\overline{14} + 5.621410$, which, divided by 7, gives $\overline{2}.803059$. Hence,

$$0.0211^{\frac{5}{7}} = 0.06354.$$

Ex. $0.0007765^{\frac{281}{170}}$.

$$\begin{array}{lr} \text{Log } 0.0007765 & \overline{4}.890141 \\ & 281 \\ \hline & \overline{4}890141 \\ & \overline{25}121128 \\ & \overline{7}780282 \\ \hline & \overline{9}26.129621 \end{array}$$

Here the third partial product, written $\overline{7}780282$ is, of course, the product of $\overline{4}. + 0.890141$ by 200. The second is the product of the same by 80, and the first by 1.

The meaning of their sum, $\overline{9}26.129621$, is $\overline{900} +$ 26.129621 or $\overline{8}74.129621$.

To divide $\overline{9}26.129621$ by 170, it will be most convenient to add and subtract -800 making $\overline{1700} +$ 826.129621. Dividing by 170,

$$
\begin{array}{lll}
170)\overline{1700} + 826.129621(\overline{10} + 4.859586 & & \\
\quad 680 & & \\
\quad \overline{1461} & & \\
\quad 1360 & & = \overline{6}.859586, \text{ whence} \\
\quad \overline{\ \ 1012} & & \\
\quad\ \ 850 & & \\
\quad\ \ \overline{1629} & & 0.0007765\tfrac{281}{170}. \\
\quad\ \ 1530 & & \\
\quad\ \ \overline{\ \ \ 996} & & \\
\quad\ \ \ \ 850 & & = 0.000007237. \\
\quad\ \ \ \ \overline{1462} & & \\
\quad\ \ \ \ 1360 & & \\
\quad\ \ \ \ \overline{\ \ 1021} & & \\
\quad\ \ \ \ \ \ 1020 & &
\end{array}
$$

The number which is added and subtracted to make the negative part of the logarithm exactly divisible may be any number whatever which will serve this purpose; but it will often be most convenient to add, as above, such number as will make

the negative part equal to ten times the divisor, because this number, and the corresponding part of the quotient can be found without any difficulty. Any other multiple, however, may be used, when the quantity to be added can be easily found. Thus, in the last example, it is evident that the addition and subtraction of 120 instead of 800, giving $\overline{1020} +$ 126.129621 would have given $\bar{6} + 0.859586$, as before.

The operations being understood, may now be more compactly arranged, as in the following example:

Ex. $0.00301^{\frac{146}{125}}$.

$$\begin{array}{lr}
\text{Log } 0.00301 & \bar{3}.478566 \\
 & 146 \\
\hline
 & \overline{16}871396 \\
 & \overline{11}914264 \\
 & \bar{3}478566 \\
\hline
 & \bar{4}31.870636
\end{array}$$

$$\begin{array}{l}
125)\overline{500} + 131.870636(\bar{4} + 1.054965 \\
\qquad\quad 125 \\
\qquad\quad \overline{687} \\
\qquad\quad625 \qquad\qquad = 3.054965 \\
\qquad\quad \overline{620} \\
\qquad\quad500 \\
\qquad\quad \overline{1206} \qquad\quad 0.001135.\text{—}\textit{Ans.} \\
\qquad\quad1125 \\
\qquad\quad \overline{813} \\
\qquad\quad750 \\
\qquad\quad \overline{636}
\end{array}$$

Ex. 428, $0.7958^{\frac{2}{3}\frac{1}{5}}$.

Ex. 429, $0.0034^{\frac{7}{3}\frac{9}{1}}$.

Ex. 430, $0.00821^{\frac{2}{3}\frac{5}{1}\frac{5}{2}}$.

Ex. 431, $0.11^{\frac{3}{1}\frac{2}{2}\frac{1}{6}}$.

Ex. 432, $0.054^{\frac{1}{3}\frac{7}{1}}$.

Ex. 433, $0.98^{\frac{3}{2}\frac{1}{5}}$.

Ex. 434, $0.101^{\frac{6}{2}\frac{5}{3}}$.

Ex. 435, $0.25^{\frac{4}{2}\frac{3}{1}}$.

Ex. 436, $0.0702^{\frac{1}{3}\frac{9}{2}\frac{1}{3}}$.

Ex. 437, $0.003^{\frac{1}{7}\frac{1}{2}}$.

(2.) Numerator of the exponent a mixed number or a decimal.

Ex. $0.00742^{\frac{1.17}{64}}$.

$$
\begin{array}{rrl}
\text{Log } 0.00742 & \bar{3}.870404 & \\
 & 1.17 & \\
\hline
 & \overline{1}5092828 & \\
 & \bar{3}870404 & \\
 & \bar{3}870404 & \\
\hline
 & \bar{3}.508373 & \\
64)\overline{64} + 61.508373(\bar{1} + 0.961068 & & \\
576 & & \\
\hline
390 & & = \bar{1}.961068 \\
384 & & \\
\hline
68 & & 0.9143.\text{—}\textit{Ans.} \\
64 & & \\
\hline
437 & & \\
384 & & \\
\hline
533 & &
\end{array}
$$

Ex. 438, $0.16^{\frac{0.71}{33}}$.

Ex. 439, $0.009^{\frac{2.5}{9}}$.

Ex. 440, $0.00735^{\frac{33.3}{71}}$.

Ex. 441, $0.804^{\frac{1.19}{55}}$.

Ex. 442, $0.511^{\frac{0.56}{83}}$.

Ex. 443, $0.0031^{\frac{0.025}{33}}$.

Ex. 444, $0.072^{\frac{0.0031}{39}}$.

Ex. 445, $0.733^{\frac{3.86}{410}}$.

Ex. 446, $0.9201^{\frac{17.4}{8}}$.

Ex. 447, $0.102^{\frac{201.1}{102}}$.

(3.) Denominator or both terms of the exponent mixed or decimal.

Ex. $0.01024^{\frac{1.33}{2.07}}$.

$$
\begin{array}{lr}
\text{Log } 0.01024 & \bar{2}.010300 \\
 & 1.33 \\
\hline
 & \bar{6}030900 \\
 & \bar{6}030900 \\
 & \bar{2}010300 \\
\hline
 & \bar{2}.653699
\end{array}
$$

$$
\begin{array}{l}
\overline{4.14} + 1.49 + 0.003699 \\
2.07)\overline{4.14} + 1.493699(\bar{2} + 0.007215 \\
\qquad\qquad 1449 \\
\qquad\qquad\;\; \overline{\;446\;} \qquad = \bar{2}.007215 \\
\qquad\qquad\;\; 414 \\
\qquad\qquad\;\;\; \overline{\;329\;} \quad 0.0101675.\text{—}\textit{Ans.} \\
\qquad\qquad\;\;\; 207 \\
\qquad\qquad\;\;\; \overline{1229} \\
\qquad\qquad\;\;\; 1035
\end{array}
$$

Ex. 448, $0.7321^{\frac{8.7}{1.09}}$.

Ex. 449, $0.0211^{\frac{11.3}{13.7}}$.

Ex. 450, $0.0247^{\frac{7.7}{6.33}}$.

Ex. 451, $0.38^{\frac{0.39}{0.71}}$.

Ex. 452, $0.0041^{\frac{1.62}{3.21}}$.

Ex. 453, $0.201^{\frac{8.8}{21.3}}$.

Ex. 454, $0.0707^{\frac{3.12}{6.24}}$.

Ex. 455, $0.63^{\frac{2.51}{1.64}}$.

Ex. 456, $0.0088^{\frac{1.81}{2.43}}$.

Ex. 457, $0.75^{\frac{3.12}{6.16}}$.

2. Exponent negative.

The method of proceeding in this case is the same as in the examples on page 46.

This case admits of the same subdivisions as the preceding one, but it will not be necessary to discuss them separately.

Ex. $2.37^{-\frac{2}{3}} = \dfrac{1}{2.37^{\frac{2}{3}}}$.

Log 2.37	0.374748
	2
	3)0.749496
	0.249832
A. C. − 10	$\bar{1}.750168$

0.56256.—*Ans.*

Ex. $0.00231^{-\frac{1.7}{3.6}}$.

$$\begin{array}{lr}
\text{Log } 0.00231 & \bar{3}.363612 \\
 & 1.7 \\
\hline
 & \bar{1}9545284 \\
 & \bar{3}363612 \\
\hline
 & \bar{4}.5181404
\end{array}$$

$$
\begin{array}{l}
3.5)\overline{7}.0 + 2.5181404(\bar{2}.719468 \\
\quad\ 245 \\
\quad\ \overline{68} \\
\quad\ 35 \qquad\qquad A.C. - 10 = 1.280532 \\
\quad\ \overline{331} \\
\quad\ 315 \\
\quad\ \overline{164} \qquad\qquad 19.078.\text{—}\textit{Ans.} \\
\quad\ 140 \\
\quad\ \overline{240} \\
\quad\ 210 \\
\quad\ \overline{304}
\end{array}
$$

Ex. 458, $1101^{-\frac{4}{9}}$.

Ex. 459, $702.2^{-\frac{16}{13}}$.

Ex. 460, $1.022^{-\frac{5}{7}}$.

Ex. 461, $20.02^{-\frac{12.1}{30.0}}$.

Ex. 462, $30.11^{-\frac{42.1}{13.3}}$.

Ex. 463, $6.1^{-3.21}$.

Ex. 464, $3.71^{-4.11}$.

Ex. 465, $7.101^{-5.33}$.

Ex. 466, $8.21^{-7.2}$.

Ex. 467, $9.021^{-1.19}$.

Ex. 468, $22^{-\frac{17.01}{11.61}}$.

Ex. 469, $1.16^{-\frac{1.25}{3.81}}$.

Ex. 470, $31.17^{-\frac{2.41}{1.30}}$.

Ex. 471, $17.41^{-\frac{1.18}{30.2}}$.

Ex. 472, $9.033^{-\frac{2.13}{3.18}}$.

Ex. 473, $0.0123^{-\frac{18.2}{74}}$.

Ex. 474, $0.00202^{-\frac{5.4}{4.3}}$.

Ex. 475, $0.00757^{\frac{3.16}{21.9}}$.

Ex. 476, $0.0255^{-\frac{11.3}{25.9}}$.

Ex. 477, $0.709^{-\frac{55.8}{49.1}}$.

II. Number negative.

The computation proceeds as in the previous case; but the result is imaginary when an even root is to be extracted.

VII. Exponential equations.

In such equations, which are of the form $a^x = b$, the exponent is the unknown quantity.

I. a and b both whole or mixed.

1. $a < b$.

Ex. $24^x = 123$.

By Principle 3, Introduction,

$$x \log 24 = \log 123$$
$$x = \frac{\log 123}{\log 24}$$

or, by Principle, 2 Introduction,

$$\operatorname{Log} x = \log \log 123 - \log \log 24.$$

$$\begin{array}{llll}
\operatorname{Log} 123 & & 2.089905 & \\
\operatorname{Log} \log 123 & = \log & 2.089905 & = 0.3201265 \\
\operatorname{Log} 24 & & 1.380211 & \\
\operatorname{Log} \log 24 & = \log & 1.380211 & = \underline{0.1399455} \\
& \operatorname{Log} x & & \ 0.1801810
\end{array}$$

$$x = 1.5142.$$

(The logarithms of logarithms can be taken most conveniently and accurately from seven-place tables.)

Ex. $33.2^x = 81$.

Log 81	1.908485	
Log log 81		0.2806889
Log 33.2	1.521138	
Log log 33.2		0.1821686
		0.0985203

$$x = 1.2547.$$

Ex. 478, $1.33^x = 5.27$. Ex. 483, $16^x = 40.2$.
Ex. 479, $1.41^x = 21$. Ex. 484, $9^x = 27.1$.
Ex. 480, $2.03^x = 10.2$. Ex. 485, $3.1^x = 11$.
Ex. 481, $4.6^x = 9$. Ex. 486, $15^x = 41$.
Ex. 482, $21.1^x = 40$. Ex. 487, $2.1^x = 3.1$.

2. $a > b$.
In this case x is obviously less than 1.

Ex. $3.19^x = 2.5$.

Log 2.5	0.397940	
Log log 2.5		$\bar{1}.5998176$
Log 3.19	0.503791	
Log log 3.19		$\bar{1}.7022504$
Log x		$\bar{1}.8975672$

$$x = 0.78989.$$

Ex. 488, $11^x = 9.2$. Ex. 493, $31^x = 9.12$.
Ex. 489, $199^x = 34.2$. Ex. 494, $11^x = 1.5$.
Ex. 490, $17.1^x = 1.81$. Ex. 495, $70^x = 40$.
Ex. 491, $42.1^x = 12.81$. Ex. 496, $11.9^x = 5$.
Ex. 492, $99^x = 20$. Ex. 497, $22^x = 19$.

II. a and b both decimals.

1. $a > b$.

Ex. $0.53^x = 0.029$.

$$x = \frac{\log 0.029}{\log 0.53} = \frac{\bar{2}.462398}{\bar{1}.724276} = \frac{-1.537602}{-0.275724}.$$

Both a and b being fractions, their logarithms are negative, and the quotient is positive. When $a > b$ x is greater than 1, and when $a < b$ x is less than 1, as it should be, since a proper fraction, to give a result less than the fraction itself, must be raised to a power whose index is greater than unity.

Neglecting the signs of the two logarithms (see page 32), we have,

$$\begin{array}{lll}
\text{Log } 0.029 = \bar{2}.462398 = -1.537602 & & \\
\qquad \text{Log log } 0.029 & & 0.1868440n \\
\text{Log } 0.53 = \bar{1}.724276 = -0.275724 & & \\
\qquad \text{Log log } 0.53 & & \underline{\bar{1}.4404743n} \\
\qquad \qquad \text{Log } x & & 0.7463697
\end{array}$$

$$x = 5.5766.$$

Ex. 498, $0.81^x = 0.02$.	Ex. 503, $0.19^x = 0.09$.
Ex. 499, $0.301^x = 0.222$.	Ex. 504, $0.333^x = 0.03$.
Ex. 500, $0.005^x = 0.0005$.	Ex. 505, $0.1^x = 0.01$.
Ex. 501, $0.171^x = 0.051$.	Ex. 506, $0.2^x = 0.0004$.
Ex. 502, $0.84^x = 0.5$.	Ex. 507, $0.13^x = 0.002$.

2. $a < b$.

Ex. $0.071^x = 0.42$.

$$
\begin{array}{lll}
\text{Log } 0.42 & \bar{1}.623249 = -0.376751 & \\
\quad \text{Log log } 0.42 \ldots\ldots\ldots\ldots\ldots\ldots & & \bar{1}.5760545n \\
\text{Log } 0.071 & \bar{2}.851258 = -1.148742 & \\
\quad \text{Log log } 0.071 \ldots\ldots\ldots\ldots\ldots\ldots & & 0.0602325n \\
\hline
\qquad \text{Log } x & & \bar{1}.5158220
\end{array}
$$

$$x = 0.32796.$$

Ex. 508, $0.02^x = 0.2$. Ex. 513, $0.21^x = 0.88$.
Ex. 509, $0.011^x = 0.41$. Ex. 514, $0.3^x = 0.9$.
Ex. 510, $0.81^x = 0.99$. Ex. 515, $0.1^x = 0.44$.
Ex. 511, $0.33^x = 0.66$. Ex. 516, $0.31^x = 0.66$.
Ex. 512, $0.02^x = 0.04$. Ex. 517, $0.5^x = 0.9$.

III. Either a or b decimal, the other whole or mixed; either a or b greater.

In this case the logarithms of a and b will have contrary signs, the quotient will be negative, and x therefore negative. This is obviously correct, since a negative power of a proper fraction will give a quantity greater than unity, and a negative power of a quantity greater than unity will give a proper fraction.

Ex. $12.4^x = 0.21$.

$$x = \frac{\log 0.21}{\log 12.4} = \frac{\bar{1}.322219}{1.093422} = \frac{-0.677781}{1.093422}, \text{ negative.}$$

Disregarding the signs, we have,

Log 0.21 $\quad \bar{1}.322219 = -0.677781.$

Log log 0.21 .... $\bar{1}.8310893n$

Log 12.4 $\quad 1.093422$

Log log 12.4 .... 0.0387878

Log x $\quad \bar{1}.7923015n$

$x = -0.61987.$

Ex. $0.029^x = 4.21.$

Log 4.21 $\quad 0.624282$

Log log 4.21 .... $\bar{1}.7953808$

Log 0.029 $\quad \bar{2}.462398 = -1.537602$

Log log 0.029 .... $0.1868440n$

Log x $\quad \bar{1}.6085368n$

$x = -0.40601.$

Ex. 518, $11.01^x = 0.9.$

Ex. 519, $7.33^x = 0.49.$

Ex. 520, $181.2^x = 0.02.$

Ex. 521, $204^x = 0.1.$

Ex. 522, $10.2^x = 0.2.$

Ex. 523, $0.21^x = 15.$

Ex. 524, $0.002^x = 1.1.$

Ex. 525, $0.12^x = 64.$

Ex. 526, $0.09^x = 90.$

Ex. 527, $0.001^x = 221.$

CHAPTER IV.

AUGMENTED LOGARITHMS.

IN many scientific works, tables of the logarithms of quantities required in calculations occur. When these quantities are fractions the characteristics of their logarithms are negative. In such cases it is usual to increase the characteristics by 10, making them thus all positive, and avoiding the use of characteristics and mantissas of contrary signs. Such logarithms may for convenience be called *augmented* logarithms. Examples of such tables are Tables III., IV., X. A, in "Chauvenet's Astronomy;" Tables II., IV., in "Bartlett's Astronomy;" Table IV. in "Bartlett's Mechanics," and all the tables of logarithms of sines, etc., to be hereafter described.

When two augmented logarithms are added together, the result is obviously greater by 20 than the true logarithm of the product of the two corresponding numbers. If m such logarithms are added and n subtracted, the result is greater by $(m - n)$ 10. In either of these cases it is only necessary to reject 2×10, or $(m - n) 10$ from the result, in order to find the true logarithm. As, however, the required number itself is also, in general, to be sought in a

table of *augmented* logarithms, one of the added tens must in this case be retained, or, in other words, only $(m - n - 1)\ 10$ must be rejected from the logarithm.

The following examples will illustrate this point:

It is required to find the number corresponding to the sum of the following logarithms, which are taken from Tables I. and II., "Bartlett's Astronomy," and of which the last two are augmented.

$$\begin{array}{lr} & 1.90440 \\ & 9.99154 \\ & 9.99709 \\ \hline \text{The true logarithm is,} & 1.89303 \end{array}$$

and the required number is 78.17.

If the required number were to be sought in a table of augmented logarithms, we should retain one of the tens, writing

$$\text{Aug. log} \ldots . = 11.89303, \text{ and finding,}$$

as before, the required number $= 78.17$.

When an augmented logarithm is multiplied by any number n, as in involution, $n \times 10$ must be rejected from the result to find the true logarithm of the required quantity, or $(n - 1)\ 10$ to find the augmented logarithm. Thus, if it is required to find the fourth power of 0.00242, we find log 0.00242 $= \bar{3}.383815$, or aug. log $= 7.383815$

$$\begin{array}{lr} & 4 \\ \hline & 29.535260 \\ \text{Rejecting } 4 \times 10, \text{ we have} & \overline{11}.535260 \end{array}$$

for the true logarithm of the required number; or, rejecting 3×10, we have $\bar{1}.535260$ for the augmented logarithm.

When an augmented logarithm is to be divided by any number n, as in evolution, the proper mode of proceeding will appear from a consideration of any of the examples in evolution. Thus (page 53), the logarithm of 0.2042 being $\bar{1}.310056$, its augmented logarithm would be 9.310056. If it is required to find the square root of 0.2042, we ought to divide $\bar{1}.310056$ by 2, finding, as before shown, $\bar{1}.655028$, the augmented logarithm corresponding to which is 9.655028. The same result would have been reached if we had added an additional ten before dividing by 2. We should then have had,

$$\begin{array}{rr} \text{true log } 0.2042 + 20 = & \\ \text{aug. log } 0.2042 + 10 = & 19.310056 \\ & \frac{1}{2} \\ \hline & 9.655028, \end{array}$$

which is the augmented logarithm of the required number.

It thus appears that, in general, before dividing by the exponent n of the root, we are to add $(n - 1)\,10$ to the augmented logarithm, or $n.10$ to the true logarithm.

Required the seventh root of the number whose true logarithm is $\bar{1}.095$. The augmented logarithm is 9.095. Adding 60, we have 69.095. Dividing by

7, we find 9.871, which is the augmented logarithm of the required number.

To the class of tables now under consideration belong the tables of logarithms of Trigonometrical Functions. Before explaining these it will be necessary to consider the nature of the functions themselves, and the use of the tables of Trigonometrical Functions.

CHAPTER V.

TABLES OF TRIGONOMETRICAL FUNCTIONS.

THE eight trigonometrical functions are all fractions, proper or improper, expressing the ratios of certain lines, which are parts of the sides of angles, or perpendiculars to one of the sides. Two of these functions, the sine and cosine are always numerically equal to, or less than, unity. Two more, the versed-sine and coversed-sine (which are seldom used in calculations) vary from 0 to 2, being never negative. The other four, the tangent, cotangent, secant, and cosecant vary from $+\infty$ to $-\infty$, and may therefore be proper or improper fractions, or whole numbers. Of these, the secant and cosecant are also little used.

A table in which the values of the sines of angles from 0° to 90° are given, is often called a table of natural sines. The word natural is unnecessary, since there is no other kind of sines, and it is better to omit it. A table of sines, then, is a series of decimal fractions, placed opposite the angles to which they belong, and increasing from 0 to 1 as the angles increase from 0° to 90°. The sines of angles

greater than 90° or less than 0° are not given, inasmuch as the sine of any such angle will have the same numerical value as the sine of some one of the angles in the table. A table of cosines is a similar series of decimal fractions, beginning with 1 and decreasing to 0 as the angle increases from 0° to 90°. The sine of an angle being the cosine of its complement, a table of sines serves also as a table of cosines. While, therefore, in some tables the sine and cosine are both printed, in parallel columns, in others the same column is made to serve both purposes, the number of degrees and minutes in the corresponding angle being printed at the *bottom* and *right* of the page for cosines, and at the *top* and *left* for sines.

A table of tangents is a series of fractions, proper and improper, increasing from 0° to 90°. For all angles less than 45° the tangent is a proper fraction, and for angles greater than 45° and less than 90° an improper fraction or a whole number. A table of tangents is also a table of cotangents, and one column of figures is often made to serve both purposes, as in the case of sines and cosines.

Tables of secants and cosecants are seldom used. Since the secant of an angle is the reciprocal of the cosine, and the cosecant the reciprocal of the sine, multiplication by a secant or cosecant is equivalent to division by a cosine or sine, and tables of secants are therefore not necessary. The secant varies from 1 to ∞, and the cosecant from ∞ to 1, as the angle

increases from 0° to 90°; so that these functions are improper fractions or whole numbers.

In any table the leading quantity, on which the other depends, is called the argument; the quantity which depends on this is the dependent variable or function. In the tables just described the angle is the argument, and the sine, etc., the dependent variable or function. In these tables the successive values of the argument usually differ from each other either by 1 minute or by 10 seconds. When a function of an angle intermediate between two angles in the table is required, or the angle corresponding to a function intermediate between two functions in the table, direct or inverse interpolation is necessary. This operation rests on the principles already explained in connection with tables of logarithms of numbers (pp. 13, 21), the only difference being that, as seconds are sixtieths of minutes, we have to deal with sixtieths of the tabular difference instead of hundredths. Thus, to find the sine of 41° 12′ 13″, in a table which gives the values for every minute we have,*

sin 41° 12′	0.658689
sin 41° 13′	0.658908
Tabular difference	219

* The examples in this chapter refer to the table of "natural" sines, etc. The tables called "logarithmic" sines will be discussed hereafter. The two must never be confounded.

Then, sin 41° 12′		0.658689
$\frac{13}{60}$ D	=	47
Therefore, sin 41° 12′ 13″	=	0.658736

If, as is usual, the tabular difference is given in the table, the above may be written more compactly,

sin 41° 12′	0.658689
$\frac{13}{60}$ D	47
sin 41° 12′ 13″	0.658736

In some tables (as Loomis') the values of $\frac{1}{60}$ D are given under the name Proportional Parts for 1″. In such cases the interpolation is still further facilitated. Thus, in the example just given, we find (Loomis' Tables) the P.P. to 1″ = 3.63. We therefore have,

sin 41° 12′	0.658689
Correction for 10″ = 10 × P.P.......	363
Correction for 3″ = 3 × P.P.......	109
sin 41° 12′ 13″	0.658736

When the argument of the table increases by 10″ instead of 1′, the principle is the same, and requires no further explanation.

In interpolating in tables of cosines and cotangents it must be remembered that as the angle increases, the function decreases ; so that, if the next smaller figure is taken from the table, the correction must be subtracted. Thus, to find

cot 42° 21′ 18″.

cot 42° 21′	1.09706 ; P.P. = 1.06
10 P.P.	− 10.6
8 P.P.	− 8.5
cot 42° 21′ 18″	1.09687

Ex. 528, sin 16° 32′ 41″. Ex. 533, sin 70° 18′ 9″.
Ex. 529, tan 34° 7′ 33″. Ex. 534, sin 89° 40′ 30″.
Ex. 530, tan 89° 58′ 12″. Ex. 535, sin 12° 0′ 17″.
Ex. 531, tan 47° 11′ 45″. Ex. 536, tan 84° 15′ 55″.
Ex. 532, sin 88° 30′ 16″. Ex. 537, tan 0° 9′ 45″.

Ex. 538, cos 44° 44′ 54″. Ex. 543, cos 75° 1′ 19″.
Ex. 539, cot 49° 18′ 31″. Ex. 544, cot 79° 21′ 34″.
Ex. 540, cot 40° 2′ 2″. Ex. 545, cot 89° 49′ 20″.
Ex. 541, cos 0° 8′ 33″. Ex. 546, cos 88° 10′ 40″.
Ex. 542, cos 11° 9′ 51″. Ex. 547, cot 61° 11′ 52″.

Inverse interpolation proceeds according to the principles already explained (p. 21), the correction to be found being a certain number of seconds, or sixtieths of a minute. It is found by dividing the difference between the given function and the next smaller function in the table by the correction for 1″. When found, it is to be added to the angle corresponding to this next smaller function in the case of sines and tangents, and subtracted from it in the case of cosines and cotangents.

The notation used is the same as that for antilogarithms. Thus, the angle whose sine is 0.713552 is written $\sin^{-1} 0.713552$.

Required $\sin^{-1} 0.553270$.

In the table we find $\sin^{-1} 0.553149 = 33° \, 35' \, 0''$

Diff. $\overline{121}$

Diff. for $1'' = 4.04$. Dividing 121 by 4.04, we find $\frac{121}{4.04} = 29.95$. Therefore, adding, $29''.95$

we find, $\sin^{-1} 0.553270 = 33° \, 35' \, 29''.95$

The division of the difference by the tabular difference (or by $\frac{1}{60}$ of it) may be extended to any number of decimal places; but, as in the corresponding operation in the table of logarithms (p. 22), the nature of the tables fixes a limit beyond which it is useless to go.

In a table of sines, $\frac{1}{60}$ D (called P.P. to $1''$) varies from 4.85 millionths for angles less than 1°, to 0.04 of a millionth at 88°. It is 1.05 millionths at 77°. For angles of about 1°, 2 tenths of a second will require in the sine a correction of 2×0.485, or 0.970 of a unit in the sixth place, and $0''.15$ will require a correction of 15×0.0485, or 0.63 of a unit in the sixth place, either of which would affect the sixth figure of the sine. In this part of the table, therefore, the angle can be determined by means of six-place sines to within $0''.15$. At 77°, since $\frac{1}{10}$ P.P. = 0.105, and $\frac{5}{10}$ P.P. = 0.525 of a millionth, no number of tenths of a second less than 5 will affect the sixth figure of the sine, and in this part of the table, therefore, the six-place sine will not determine the angle

more nearly than to within $0''.5$. In the last part of the table, P.P. being 0.04 of a millionth, a difference of $13''$ in the angle will produce only a difference of 0.52 of a unit in the sixth place in the sine, and the angle cannot be determined more nearly than to within $13''$. It appears, therefore, that in inverse interpolation it is useless to carry the operation beyond seconds in the last part of the table, or beyond hundredths of seconds in the first part.

In the table of cosines, on the other hand, the differences are greatest for large angles and least for small ones. Large angles are therefore determined most accurately by their cosines, and small angles by their sines.

The value of $\frac{1}{60}$ D for tangents is 4.85 millionths for angles less than $2°$, 9.53 for angles between $44°$ and $45°$, 0.99 between $45°$ and $46°$, 13.01 at $86°$, and 1718.88 at $89°\ 59'$. Angles therefore can be determined by their tangents, within $0''.15$ in the first part of the table, within $0''.06$ just before $45°$, within $0.5''$ just above $45°$, and within $0''.04$ at $86°$. In the table of cotangents, on the other hand, the corresponding degrees of precision are found at the complementary angles.

Ex. 548, $\sin^{-1}$ 0.319752.	Ex. 553, $\sin^{-1}$ 0.000722.
Ex. 549, $\cos^{-1}$ 0.549777.	Ex. 554, $\sin^{-1}$ 0.999987.
Ex. 550, $\sin^{-1}$ 0.998144.	Ex. 555, $\cos^{-1}$ 0.999996.
Ex. 551, $\cos^{-1}$ 0.014345.	Ex. 556, $\cos^{-1}$ 0.345567.
Ex. 552, $\cos^{-1}$ 0.000599.	Ex. 557, $\cos^{-1}$ 0.712466.

Ex. 558, $\tan^{-1} 0.592994$.
Ex. 559, $\tan^{-1} 2.537496$.
Ex. 560, $\cot^{-1} 1.895432$.
Ex. 561, $\tan^{-1} 0.983150$.
Ex. 562, $\cot^{-1} 0.999401$.
Ex. 563, $\tan^{-1} 32.1195$.
Ex. 564, $\cot^{-1} 17.1540$.
Ex. 565, $\cot^{-1} 289.3127$.
Ex. 566, $\cot^{-1} 0.089344$.
Ex. 567, $\tan^{-1} 0.066588$.

The functions of angles greater than 90° are not contained in the tables, but are readily found from the relations which exist between them and those of certain angles less than 90°. Of these relations, the most convenient are the following:

1. To find the functions of an angle in the 2d quadrant.

$$\sin(180° - a) = \sin a.$$
$$\cos(180° - a) = -\cos a.$$
$$\tan(180° - a) = -\tan a.$$
$$\cot(180° - a) = -\cot a.$$

2. To find the functions of an angle in the 3d quadrant.

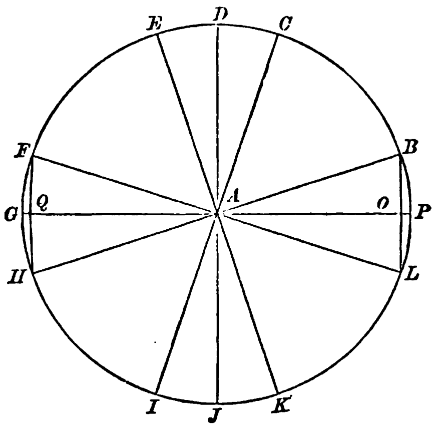
$$\sin(180° + a) = -\sin a.$$
$$\cos(180° + a) = -\cos a.$$
$$\tan(180° + a) = \tan a.$$
$$\cot(180° + a) = \cot a.$$

3. To find the functions of an angle in the 4th quadrant.

$$\sin(360° - a) = -\sin a.$$
$$\cos(360° - a) = \cos a.$$
$$\tan(360° - a) = -\tan a.$$
$$\cot(360° - a) = -\cot a.$$

It is in general not best to try to remember these formulas; but they are readily recalled by the aid of the diagram below, in which, if the angle PAB be called A, we have

$$
\begin{aligned}
PAC &= 90^\circ - A,\\
PAE &= 90^\circ + A,\\
PAF &= 180^\circ - A,\\
PAH &= 180^\circ + A,\\
PAI &= 270^\circ - A,\\
PAK &= 270^\circ + A,\\
PAL &= 360^\circ - A,
\end{aligned}
$$



the last four angles being greater than 180°. The directions and magnitudes of the lines OB, QF, QH, and OL will readily recall the first formula in each of the three groups on the preceding page. The lines AO and AQ will recall the second formulas.

The third and fourth may be recalled by appropriate lines, or deduced from the first and second by the considerations that $\tan a = \frac{\sin a}{\cos a}$ and $\cot a = \frac{\cos a}{\sin a}$.

Ex. Required sin 195° 10′.

$$\text{Sin } 195° 10' = \sin (180° + 15° 10') = -\sin 15° 10'.$$

Hence, from the table, we find

$$\sin 195° 10' = -0.261628.$$

Similarly we find

$$\begin{aligned} \tan 295° 12' 10'' &= \tan (360° - 64° 47' 50''). \\ &= -\tan 64° 47' 50'' = -2.12481 \\ \cos 301° 10' 15'' &= \cos (360° - 58° 49' 45''). \\ &= \cos 58° 49' 45'' = 0.517592. \end{aligned}$$

Ex. 568, sin 110° 4′ 24″.
Ex. 569, cos 168° 13′ 7″.
Ex. 570, cot 177° 2′.
Ex. 571, tan 109° 31′ 30″.
Ex. 572, cot 259° 16′.
Ex. 573, sin 264° 7′ 17″.
Ex. 574, cot 301° 16′.
Ex. 575, sin 299° 12′ 25″.
Ex. 576, cos 322° 6′ 1″.
Ex. 577, tan 275° 5′.

It is now evident that, when an angle is determined by means of one of its functions the result will in general be ambiguous. Thus, a given *sine*, if positive, may belong either to an angle A or to 180° − A; if negative, to 180° + A or 360° − A. A given cosine, if positive, belongs to A or 360° − A; if negative, to 180° − A or 180° + A. A given tan-

gent, if positive, belongs to A or $180° + A$; if negative, to $180° - A$ or $360° - A$. A given cotangent, if positive, belongs to A or $180° + A$; if negative, to $180° - A$ or $360° - A$.

Unless, therefore, we have other means than the value of the function, we cannot completely determine the angle, but can at best only determine that it is one of two certain angles. In the problems of plane trigonometry, however, the angles are always less than 180°, and all the ambiguities just pointed out therefore disappear, excepting the one which results from the equality of sin A and $\sin(180° - A)$. To avoid this it is usual to determine the angle, when possible, by means of its cosine, tangent, or cotangent, since no two angles less than 180° have these functions the same.

Ex. Required $\sin^{-1} - 0.271069$.

The sine being negative, the angle is greater than 180° and less than 360°. If $a = \sin^{-1} 0.271069$, we have $\sin^{-1} - 0.271069$, either $180° + a$ or $360° - a$. We find in the table,

$$\begin{aligned} \sin^{-1} \quad 0.271069 &= \ \ 15° \, 43' \, 40''.5. \quad \text{Hence,} \\ \sin^{-1} - 0.271069 &= 195° \, 43' \, 40''.5; \text{ or,} \\ & \quad\ 344° \, 16' \, 19''.5. \end{aligned}$$

Ex. Required $\tan^{-1} - 2.12472$.

$$\begin{aligned} \tan^{-1} - 2.12472 &= 180° - \tan^{-1} 2.12472, \\ & \text{or } 360° - \tan^{-1} 2.12472. \end{aligned}$$

Hence, from the table,

$$\tan^{-1} - 2.12472 = 180° - 64° 47' 46''.6 = 115° 12' 13''.4,$$
$$\text{or } 360° - 64° 47' 46''.6 = 295° 12' 13''.4.$$

Ex. $\cos^{-1} 0.517585 = 58° 49' 46''.5,$
$$\text{or } 360° - 58° 49' 46''.5 = 301° 10' 13''.5.$$

Ex. 578, $\cos^{-1}$ 0.712555.	Ex. 583, $\cot^{-1} - 2.47561$.
Ex. 579, $\tan^{-1}$ 21.4926.	Ex. 584, $\cos^{-1} - 0.882700$.
Ex. 580, $\sin^{-1} - 0.499622$.	Ex. 585, $\cos^{-1}$ 0.516538.
Ex. 581, $\cos^{-1} - 0.884337$.	Ex. 586, $\sin^{-1}$ 0.331329.
Ex. 582, $\tan^{-1} - 11.7421$.	Ex. 587, $\sin^{-1} - 0.411704$.

All the operations of practical triogonometry can be performed with the tables of trigonometrical functions without the aid of logarithms. Thus,

Given the base b of a right-angled triangle, 235 feet, and the angle at the base, A, 31° 10′ 21″, to find the perpendicular, a.

We have $a = b \tan A$. From the table we find, $\tan 31° 10' 21'' = 0.604967$, and multiplying by 235, we find $a = 142^{f}.17$.

Again,

Given the angles A and B of an oblique-angled triangle, 41° 12′ 3″ and 66° 18′ 1″, and the side a, $719^{f}.2$, to find b.

We have, $\frac{b}{a} = \frac{\sin B}{\sin A}$, hence $b = a \frac{\sin B}{\sin A}$,

$$\text{or, } b = 719.2 \frac{0.915665}{0.658700} = 999^{f}.7.$$

Examples. Right-angled triangles ($C = 90°$).

Ex. 588, $A = 5° 16' 44''$, $c = 319.2$ ft.

Ex. 589, $B = 64° 11' 19''$, $a = 41.4$ ft.

Ex. 590, $A = 84° 6' 10''$, $b = 1600$ ft.

Ex. 591, $A = 70° 10' 22''$, $b = 121$ ft.

Ex. 592, $B = 5° 1' 10''$, $c = 1421$ ft.

Ex. 593, $a = 74.1$ ft., $c = 333.3$ ft.

Ex. 594, $a = 51.5$ ft., $b = 17.42$ ft.

Ex. 595, $b = 111.3$ ft., $c = 291$ ft.

Examples. Oblique-angled triangles.

Ex. 596, $A = 41° 9' 11''$, $B = 66° 55' 44''$, $b = 266$ ft.

Ex. 597, $A = 16° 34' 50''$, $B = 49° 18' 57''$, $b = 22.1$ ft.

Ex. 598, $a = 33$ ft., $b = 74$ ft., $A = 20° 17' 14''$.

Ex. 599, $a = 144$ ft., $b = 217$ ft., $C = 29° 19' 25.''$

(In the last example, the formula $\tan \frac{1}{2}(B - A) = \tan \frac{1}{2}(B + A) \frac{b - a}{b + a}$ will make known $\frac{1}{2}(B - A)$, and this, with $\frac{1}{2}(B + A)$, will determine B and A.)

The operations of multiplication and division in these examples are greatly facilitated by the use of logarithms. Thus, returning to the first example solved above, we find,

$$\log a = \log b + \log \tan A.$$
$$= \log 235 + \log 0.604967.$$

Log 235	2.371068
Log 0.604967	$\bar{1}.781731$
	2.152799

$$a = 142.17.$$

The other examples may be solved in the same way.

CHAPTER VI.

TABLES OF LOGARITHMS OF TRIGONOMETRICAL FUNCTIONS.

THE use of the tables thus far described, however, requires for each angle, that we find first the sine or other function of the angle in a table of sines, etc., and second the logarithm of this number in a table of logarithms. These two operations are reduced to one by the use of a table in which, instead of the sines, tangents, etc., the logarithms of these numbers are given opposite the angles. Such a table is commonly, but improperly, called a table of *logarithmic sines*, etc. It should be called a table of *logarithms of sines*, etc. Such a table will always be used whenever the operation of multiplication, division, involution or evolution is to be performed on any of the trigonometrical functions. The table of sines, themselves, (often called *natural* sines), will be used only when operations of addition and subtraction are to be performed, which cannot be performed by means of logarithms. The two tables must not be confounded.

The manner of using these tables will be easily understood, after the explanations that have already been given of the use of tables of logarithms.

The only points requiring particular attention are,

1st, Interpolation; 2d, The correction sometimes needed on account of the augmentation of the logarithms.

INTERPOLATION.

1. Direct interpolation.

When the logarithm of some function of an angle not contained in the table is required, it is only necessary to add to the corresponding function of the next smaller angle in the table, the proper proportional part of the tabular difference. If the angles in the table increase by 1′, the correction for n seconds is n sixtieths of the tabular difference. If the angles increase by 10″, the correction is n tenths of the difference. In either case, the calculation is facilitated by means of a table of "Proportional Parts to 1″," (as in Loomis' tables), which gives the value of the correction for 1″.

Required the logarithm of the sine of 40° 29′ 13″ (written log sin 40° 29′ 13″).

We find, in the table,

Log sin 40° 29′ 10″	9.812421
Log sin 40° 29′ 20″	9.812446
Tab. diff.	25
$\frac{3}{10}$ diff.	7.5
which, added to log sin 40° 29′ 10″	9.812421
gives log sin 40° 29′ 13″	9.812428

Using the proportional parts at the bottom of the page (Loomis' tables), we find

Log sin 40° 29′ 10″	9.812421
P.P. for 3″	7
Log sin 40° 29′ 13″	9.812428

It must be remembered that this is the augmented or tabular logarithm. The true logarithm of the sine of 40° 29′ 13″ is $\bar{1}.812428$

Required log tan 54° 2′ 16″.

Log tan 54° 2′ 10″		10.139315
P.P. for 6″		27
Aug. log	=	10.139342.—*Ans.*

Required log tan 89° 45′ 54″.

Log tan 89° 45′ 50″		12.385004
P.P. for 4″		1996.8
Aug. log	=	12.387001.—*Ans.*

The same table gives the logarithms of the cosines and cotangents of angles, the degrees and minutes being given in this case at the *bottom* and *right* of the page, instead of the *top* and *left*. In some tables (as Davies') sines, cosines, tangents and cotangents are given in separate columns, the figures at the top and left of the page being used, in this case, only for angles less than 45°, and those at the bottom and right, for angles greater than 45°.

In interpolating in the table of logarithms of cosines or of cotangents, it must be remembered that the correction for seconds is to be *subtracted*, if the function of the next less angle is taken. If the next larger angle is used, the correction must be added.

Required log cos 68° 26′ 39″.

Log cos 68° 26′ 30″	9.565196
P.P. for 9″ (subtracting)	48
	9.565148.—*Ans.*

Required log cot 53° 51′ 54″.

Log cot 53° 51′ 50″	9.863429
P.P. for 4″	18
	9.863411.—*Ans.*

Ex. 600, log sin 3° 1′ 12″
Ex. 601, log tan 21° 41′ 37″
Ex. 602, log cos 68° 9′ 26″
Ex. 603, log cot 84° 50′ 16″
Ex. 604, log cot 83° 12′ 0″ *
Ex. 605, log sin 89° 23′ 14″
Ex. 606, log tan 89° 25′ 16″
Ex. 607, log cos 1° 15′ 15″
Ex. 608, log cot 1° 44′ 42″
Ex. 609, log cot 5° 22′ 0″

For angles less than 2° or greater than 88°, auxiliary tables are sometimes used, which give the logarithms of the sines, etc., more accurately than

* Given in Loomis' Tables as cot 83° 11′ 60″.

the ordinary tables. Such auxiliary tables are given in Loomis' tables, and their use is there sufficiently explained. They will not be needed, except when great accuracy is required.*

2. Inverse interpolation.

The next smaller logarithm in the table than the given logarithm being found, and the corresponding angle taken, this angle is to be increased (or diminished) by the number of seconds obtained by dividing the difference between the given logarithm and the next smaller one by the difference for one second. This division may be performed in the ordinary way, or by means of the Proportional Parts, as in the table of logarithms of numbers.

Required the angle of which the logarithm of the sine is 9.057222 (written, log $\sin^{-1}$ 9.057222). (This number is, of course, the augmented log sin.) We find,

Nearest log sin, 9.057172		$6° \ 33' \ 0''$
Diff.,	50; and $\frac{50}{18.5} =$	$2''.7$
Hence, log $\sin^{-1}$ 9.057222		$6° \ 33' \ 2''.7$

By Proportional Parts,

Nearest log sin, 9.057172 = log sin 6° 33′ 0″	
Diff. = 50. Correction for 37	$2''$
$50 - 37 = 13$. " " 12.9	$0''.7$
Ans.	$6° \ 33' \ 2''.7$

* See Loomis' Tables, pp. 114, 115.

Required log $\tan^{-1}$ 9.638784.

Nearest log tan	9.638762 =	log tan	23° 31′ 20″
	22.		
	17 =	correction for	3″
22 − 17 = 5.	4.6 =	" "	0″.8
5 − 4.6 = 0.4.	0.4 =	" "	0″.07
		Ans.	23° 31′ 23″.87

Required log $\cos^{-1}$ 9.508294.

	9.508276 =	log cos	71° 11′ 50″
Diff.,	18.	13	2″
		5	0″.8
Subtracting, we find		*Ans.*	71° 11′ 47″.2

Required log $\cot^{-1}$ 10.300255.

	210 =	log cot	26° 36′ 30″
Diff.,	45.	42	8″
		3	0″.6
		Ans.	26° 36′ 21″.4

Ex. 610, log $\sin^{-1}$ 9.617843
Ex. 611, log $\tan^{-1}$ 9.820009
Ex. 612, log $\sin^{-1}$ 9.720024
Ex. 613, log $\tan^{-1}$ 9.920039
Ex. 614, log $\sin^{-1}$ 8.194732
Ex. 615, log $\cos^{-1}$ 9.860020
Ex. 616, log $\cot^{-1}$ 10.160066
Ex. 617, log $\cot^{-1}$ 11.715964
Ex. 618, log $\cos^{-1}$ 9.999992
Ex. 619, log $\cos^{-1}$ 9.996424

It has been shown (p. 85), how the functions of angles greater than 90° may be found. The same principles apply in finding the logarithms of these functions. Thus,

$$\log \sin 161^\circ\ 10' = \log \sin (180^\circ - 161^\circ\ 10') = \log \sin 18^\circ\ 50' = 9.508956.$$

In those cases in which the function is negative, it must be remembered that, as shown on page 32, the logarithm is to be taken as if the function were positive. To indicate that the quantity is negative, the letter n is written after the logarithm; and when several such functions enter into the same formula, the sign of the result depends on the number of such negative factors.

$$\begin{aligned}
\text{Log} \cos 144^\circ\ 34' &= \log(-\cos 35^\circ\ 26') \\
&= 9.911046n \\
\text{Log} \tan 98^\circ\ 10' &= \log(-\tan 81^\circ\ 50') \\
&= 10.843123n \\
\text{Log} \cot 116^\circ\ 30'\ 20'' &= \log(-\cot 63^\circ\ 29'\ 40'') \\
&= 9.697842n \\
\text{Log} \cos 224^\circ\ 45'\ 5'' &= \log(-\cos 44^\circ\ 45'\ 5'') \\
&= 9.851361n
\end{aligned}$$

Ex. 620, log cos 301° 10′
Ex. 621, log cot 144° 54′
Ex. 622, log tan 230° 6′
Ex. 623, log sin 235° 12′
Ex. 624, log cos 250° 10′ 20″
Ex. 625, log sin 121° 16′ 4″
Ex. 626, log sin 155° 2′ 31″

Ex. 627, $\log \sin 276° \ 3' \ 18''$
Ex. 628, $\log \cot 188° \ 17'$
Ex. 629, $\log \cos 194° \ 16' \ 3''$

In finding the angle from the logarithm of the function, the same ambiguity occurs as when it is found from the function itself (see p. 87), and the same means must be used to remove the ambiguity.

Ex. $\log \sin^{-1} 9.552456n$
$= 180° + \log \sin^{-1} 9.552456$;
or, $360° - \log \sin^{-1} 9.552456$
$= 180° + 20° \ 54' \ 19''$ or, $360° - 20° \ 54' \ 19''$
$= 200° \ 54' \ 19''$ or, $339° \ 5' \ 41''$

Ex. $\log \tan^{-1} 10.005099n$
$= 180° - \log \tan^{-1} 10.005099$;
or, $360° - \log \tan^{-1} 10.005099$
$= 180° - 45° \ 20' \ 11''$ or, $360° - 45° \ 20' \ 11''$
$= 134° \ 39' \ 49''$ or, $314° \ 39' \ 49''$

Ex. 630, $\log \tan^{-1} 11.432290n$
Ex. 631, $\log \sin^{-1} 9.123345$
Ex. 632, $\log \cos^{-1} 9.461132$
Ex. 633, $\log \cot^{-1} 10.471621n$
Ex. 634, $\log \cos^{-1} 8.307290$
Ex. 635, $\log \tan^{-1} 8.250120n$
Ex. 636, $\log \sin^{-1} 8.417174n$
Ex. 637, $\log \cot^{-1} 10.717123n$
Ex. 638, $\log \tan^{-1} 9.610520$
Ex. 639, $\log \cos^{-1} 9.381470n$

These ambiguities, except that of the sine, will seldom occur in practice.

The correction for 1″ in the table of logarithms of sines varies from 689.4 units in the sixth place for an angle of 10′, to 1 unit for angles between 55° and 76°, and at 88° it is only 0.1. In the first part of the table a difference of $\frac{1''}{689}$ in the angle will affect the sixth figure of the log sin, and in the last part no difference less than 10″ will affect it. Angles therefore can be determined by the logarithms of their sines, within $\frac{1''}{689}$ in the first part of the table, 1″ in the part between 55° and 76°, and 10″ at 88°. The logarithms of cosines determine angles with the corresponding degrees of precision in the complementary parts of the table.

The tabular difference in the table of log tans increasing more rapidly, the logarithms of the tangents determine angles, in the last part of the table more exactly than the logarithms of the sines.

AUGMENTATION OF LOGARITHMS OF TRIGONOMETRICAL FUNCTIONS.

The sines and cosines of angles being proper fractions, their logarithms have negative characteristics. It is usual therefore (see p. 74) to increase these logarithms by 10; and in using a table of such logarithms the precautions given in Chapter IV, must be heeded. As tangents and cotangents are also, in

part, proper fractions, their logarithms are also augmented. Accordingly, when the *true* logarithm of a quantity is to be found by means of a formula containing m trigonometrical functions as multipliers and n such functions as divisors $(m-n)\,10$ must be subtracted from the sum of the logarithms, and when the augmented logarithm is required $(m-n-1)\,10$ must be subtracted. Thus,

(Ex. 1, page 89,)

$a = b \tan A$; or, $a = 235 \tan 31^\circ\ 10'\ 21''$.

Log 235	2.371068
Aug. log tan $31^\circ\ 10'\ 21''$	9.781731
True log a	2.152799

$$a = 142.17.$$

Ex. $c = 441.2$, $b = 283.05$, $C = 90^\circ$; to find B.

We have, by plane trigonometry, $\sin B = \dfrac{b}{c}$.

Hence,

Log 283.05	2.451862
Log 441.2	2.644636
Log sin B	$\bar{1}.807226$
Aug. log sin B	9.807226

$$B = 39^\circ\ 54'\ 25''.$$

When two sides of a right-angled triangle are given, the third is found, without the trigonometrical tables, by principles 3 and 4, Introduction. The computation may be arranged as follows:

Ex. $a = 42.1$, $b = 184.1$, $C = 90°$; to find c.

$$
\begin{array}{lr}
\text{Log } 42.1 & 1.624282 \\
 & 2 \\
\hline
\text{Log } a^2 & 3.248564 \\
\quad a^2 = 1772.4 & \\
\text{Log } 184.1 & 2.265054 \\
 & 2 \\
\hline
\text{Log } b^2 & 4.530108 \\
\quad b^2 = 33893.1 & \\
a^2 + b^2 = c^2 = 35665.5 & \\
\text{Log } c^2 & 4.552248 \\
 & \frac{1}{2} \\
\hline
\text{Log } c & 2.276124 \\
\quad c = 188.8 &
\end{array}
$$

Ex. 2, page 89. (Oblique-angled triangle.)

$$b = a\frac{\sin B}{\sin A} = 719.2\frac{\sin 66° \ 18' \ 1''}{\sin 41° \ 12' \ 3''}$$

$$
\begin{array}{lr}
\text{Log } 719.2 & 2.856850 \\
\text{Aug. log sin } 66° \ 18' \ 1'' & 9.961736 \\
\text{Aug. log sin } 41° \ 12' \ 3'' \text{ (A. C.)} & 0.181312 \\
\hline
 & 2.999898 \\
\quad b = 999.7 &
\end{array}
$$

If, in such a case as the last, an angle (as A) were required instead of a side, we should have

$$\sin A = \sin B\,\frac{a}{b}; \text{ or,}$$

$$\text{Aug. log sin } A = \text{Aug. log sin } B + \log a + (\text{A. C. } \log b - 10),$$

and since A is to be determined by means of the *augmented* logarithm of its sine, we do not reject a ten on account of the use of the aug. log sin B ; that is, instead of rejecting $m \times 10$, as we should if we required the *true* log sin A, we reject $(m - 1)\,10$ or 0, to find the aug. log sin A.

Given, $a = 41.7$, $b = 98.9$, $B = 63°\ 11'$, find A.

$$\sin A = \sin B \frac{a}{b}.$$

Aug. log sin B	9.950586
Log a	1.620136
Colog b	8.004804
Aug. log sin A	9.575526, rejecting 10

on account of the arithmetical complement used, but not for the augmented logarithm, since an augmented logarithm is required. Hence,

$$A = 22°\ 6'\ 15''.$$

Given $a = 97$, $c = 421$, $B = 61°\ 12'$. Required A and C.

$$\frac{c + a}{c - a} = \frac{\tan \frac{1}{2}(C + A)}{\tan \frac{1}{2}(C - A)}.\quad \text{Hence, } \tan \tfrac{1}{2}(C - A) =$$

$$\tan \tfrac{1}{2}(C + A)\,\frac{c - a}{c + a}.$$

Log tan $\frac{1}{2}$ (C + A)	= log tan 59° 24′	10.228120
Log $(c - a)$	= log 324	2.510545
Colog $(c + a)$	= colog 518	7.285670
		10.024335

Retaining one of the two added tens because an augmented logarithm is required, we find,

$$
\begin{aligned}
\tfrac{1}{2}(C - A) &= 46^\circ\ 36'\ 16''; \text{ and since} \\
\tfrac{1}{2}(C + A) &= 59^\circ\ 24' \\
\hline
C &= 106^\circ\ \ 0'\ 16'' \\
A &= 12^\circ\ 47'\ 44''
\end{aligned}
$$

The remaining side is found as in Ex. 2, p. 102.

Given $a = 29.2$, $b = 36.5$, $c = 21$. Required the angles.

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(S - b)(S - c)}{b\,c}}, \text{ where } S = \frac{a + b + c}{2}.$$

Here, $S = 43.35$, $S - b = 6.85$, $S - c = 22.35$.

Log 6.85	0.835691
Log 22.35	1.349278
Colog 36.5	8.437707
Colog 21.	8.677781

Hence, *true* $\log \sin^2 \tfrac{1}{2} A = \bar{1}.300457$
$= \bar{2} + 1.300457$ (see p. 53.)

Dividing by 2, we find

true $\log \sin \tfrac{1}{2} A = \bar{1}.650228$

and therefore, *aug.* " " $= 9.650228$.

The same result would have been found if we had added 2×10 before dividing by 2 (see p. 76).

Hence we find $\tfrac{1}{2} A = 26^\circ\ 32'\ 46''$
$A = 53^\circ\ \ 5'\ 32''$.

The other angles are then easily found.

Examples, $C = 90$.

Ex. 640, $c = 40.12$, $b = 21.56$.
Ex. 641, $c = 901.11$, $A = 39° 12' 19''$.
Ex. 642, $c = 70.04$, $a = 32.19$.
Ex. 643, $c = 1601$, $B = 88° 10' 28''$.
Ex. 644, $a = 20.4$, $b = 449.8$.
Ex. 645, $a = 49.8$, $b = 55.4$.

Examples of oblique-angled triangles.

Ex. 646, $A = 59° 9' 14''$, $B = 12° 16' 10''$, $c = 74$.
Ex. 647, $A = 121° 4' 21''$, $B = 30° 8' 5''$, $c = 112.2$.
Ex. 648, $A = 95° 14' 17''$, $B = 64° 2' 20''$, $a = 412$.
Ex. 649, $a = 311.2$, $b = 41.9$, $A = 24° 11' 5''$.
Ex. 650, $a = 1724.5$, $b = 89.3$, $A = 111° 2' 41''$.
Ex. 651, $a = 39.25$, $b = 55.4$, $C = 38° 12' 13''$.
Ex. 652, $a = 427.1$, $b = 580.02$, $C = 98° 4' 15''$.
Ex. 653, $a = 55.1$, $b = 47.9$, $c = 88.8$.
Ex. 654, $a = 1029.$, $b = 32.7$, $c = 1020.1$.
Ex. 655, $a = 671.$, $b = 942.7$, $c = 1529$.

SPHERICAL TRIGONOMETRY.

Right-angled triangles.

When one of the parts of a triangle is found by means of Napier's rules for circular parts, we always have either—

1. The sine or cosine of the unknown part equal to the product of two functions of the given parts, or,

2. The sine or cosine of the unknown part equal to the ratio of two functions of the given parts.

In the first case, in logarithmic computation, we have the sum of two augmented logarithms, and have therefore to reject 10, since, the unknown quantity being an angle, an augmented logarithm is required.

In the second case, we have the difference of two augmented logarithms, the added tens disappear, and 10 must be added.

The following example will illustrate both cases.

Given $a = 41° 12'$, $b = 29° 16' 10''$, $(C = 90°)$.

1. $\cos c = \cos a \cos b$.

Log cos 41° 12′	9.876457
Log cos 29° 16′ 10″	9.940681
	9.817138

$c = 48° 58' 39''.5$.

2. $\sin b = \tan a \cot A \therefore \cot A = \frac{\sin b}{\tan a}$.

Log sin 29° 16′ 10″	9.689235
Log tan 41° 12′	9.942223
True log cot A	$\bar{1}.747012$
Aug. log cot A	9.747012

$A = 60° 49' 2''$.

Ex. 656, $c = 116° \ 21' \ 26''$, $A = 69° \ 16' \ 30''$.
Ex. 657, $c = 149° \ 15'$, $a = 26° \ 21'$.
Ex. 658, $B = 159° \ 10' \ 10''$, $b = 166° \ 20' \ 15''$.
Ex. 659, $B = 134° \ 0' \ 1''$, $a = 122° \ 0' \ 10''$.
Ex. 660, $a = 46° \ 0' \ 3''.4$, $b = 66° \ 7'$.
Ex. 661, $A = 69° \ 7' \ 4''.5$, $B = 37° \ 46' \ 0''.2$.

(For methods of checking or verifying the computation, see "Chauvenet's Trigonometry.")

Oblique-angled triangles.

All the cases of oblique-angled spherical triangles may be solved by means of the following formulas, though others are often used, which will be found in Chauvenet and other works on trigonometry. If the logarithmic solution of these formulas is well understood, no others will present any difficulty.

(1.) $$\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}.$$

(2.) $$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (S - b) \sin (S - c)}{\sin b \sin c}}.$$

(3.) $$\sin \tfrac{1}{2} a = \sqrt{\frac{- \cos S \cos (S - A)}{\sin B \sin C}}.$$

(4.) $$\frac{\cos \frac{1}{2} (a + b)}{\cos \frac{1}{2} (a - b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A + B)};$$

$$\frac{\sin \frac{1}{2} (a + b)}{\sin \frac{1}{2} (a - b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A - B)}.$$

(5.) $\dfrac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} = \dfrac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a+b)}$;

$\dfrac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} = \dfrac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a-b)}$.

Ex. Given $b = 114° \; 2' \; 31''$, $a = 39° \; 21' \; 16''$, $C = 44° \; 10'$.
We find $b + a = 153° \; 23' \; 47''$; $b - a = 74° \; 41' \; 15''$, hence $\frac{1}{2}(b+a) = 76° \; 41' \; 53''.5$; $\frac{1}{2}(b-a) = 37° \; 20' \; 37''.5$ and $\frac{1}{2} C = 22° \; 5'$. Hence, by formulas (4),

Log cot	22° 5′	10.391775	
Log cos	37° 20′ 37″.5	9.900373	
Colog cos	76° 41′ 53″.5	0.638119	
Aug. log tan $\frac{1}{2}$ (A + B)		10.930267	
$\frac{1}{2}$ (A + B)			83° 18′ 11″.4

Log cot	22° 5′	10.391775	
Log sin	37° 20′ 37″.5	9.782899	
Colog sin	76° 41′ 53″.5	0.011810	
Aug. log tan $\frac{1}{2}$ (B − A)		10.186484	
$\frac{1}{2}$ (B − A)			56° 56′ 23″.5
		B =	140° 14′ 34″.9
		A =	26° 21′ 47″.9

The side c may be found from formula (1).

A little consideration will make it appear that, in the addition above, only two tens have been added, and therefore one is retained, in order to find the augmented logarithm.

In using formula 1, as, for instance, to find $\sin c = \sin a \dfrac{\sin C}{\sin A}$, if we add the cologarithm of the third quantity, sin A, since this is then not augmented, only two augmented logarithms are used, and only one ten is to be rejected from the result.

Cases involving formulas (5), are treated similarly to the last. The following example will illustrate the use of formula (3), and that of (2) will be similar.

$$\text{Ex. Given} \left\{\begin{array}{l} A = 41^\circ\ 10'\ 10'' \\ B = 155^\circ\ 11'\ 6'' \\ C = \underline{21^\circ\ 4'\ 4''} \end{array}\right\}$$

$$\begin{array}{lrl} \text{We find} & S = \tfrac{1}{2} & (217^\circ\ 25'\ 20'' \\ & = & 108^\circ\ 42'\ 40'' \\ \text{Hence,} & S - A = & 67^\circ\ 32'\ 30'' \\ & S - B = - & 46^\circ\ 28'\ 26'' \\ & S - C = & 87^\circ\ 38'\ 36'' \end{array}$$

Also, $\cos S = -\cos\,(180^\circ - S) = -\cos 71^\circ\ 17'\ 20''$
and therefore $-\cos S = \qquad \cos 71^\circ\ 17'\ 20''$

Log cos $71^\circ 17' 20''$		9.506230
Log cos $67^\circ 32' 30''$		9.582076
Colog sin $155^\circ 11' 16''$ = colog sin $24^\circ 48' 44''$		0.377117
Colog sin $21^\circ\ 4' 4''$		0.444336
Aug. log $\sin^2 \frac{1}{2} a$		19.909759
		$\frac{1}{2}$
Aug. log $\sin \frac{1}{2} a$		9.954879
$\frac{1}{2} a$	$64^\circ\ 19'\ 56''$	
a	$128^\circ\ 39'\ 52''$	

Here the third and fourth logarithms being arithmetical complements, are not augmented, as the added tens disappear in subtracting from ten. The first two logarithms are augmented, and twenty should therefore be rejected from the result, to find the true log $\sin^2 \frac{1}{2} a$. But (p. 76), two tens are added to make the result divisible by two and obtain the augmented logarithm of $\sin \frac{1}{2} a$. Similarly b and c may be found.

Ex. 662, $a = 43° 17'$, $b = 29° 16'$, $A = 54° 16' 10''$.

Ex. 663, $a = 124° 10'$, $c = 132° 11'$, $C = 166° 9' 12''$.

Ex. 664, $a = 99° 3' 4''$, $b = 81° 16' 44''$, $B = 55° 9' 9''$.

Ex. 665, $b = 84° 5' 12''$, $c = 39° 7' 51''$, $B = 144° 19'$.

Ex. 666, $b = 174° 0' 13''$, $c = 81° 7' 27''$, $C = 63° 10'$.

Ex. 667, $A = 5° 5' 13''$, $B = 166° 19'$, $b = 93° 7'$.

Ex. 668, $A = 88° 17'$, $B = 71° 4'$, $a = 67° 2' 3''$.

Ex. 669, $B = 84° 10' 3''$, $C = 5° 19'$, $c = 8° 4' 2''$.

Ex. 670, $B = 84° 10' 3''$, $C = 56° 11' 1''$, $c = 41° 14' 9''$.

Ex. 671, $B = 93° 19' 11''$, $C = 64° 7'$, $b = 88° 17'$.

Ex. 672, $a = 45° 29' 4''$, $b = 24° 9' 17''$, $c = 68° 10'$.

Ex. 673, $a = 84° 11' 51''$, $b = 101° 6'$, $c = 140° 17'$.

Ex. 674, $a = 104° 4'$, $b = 121° 16'$, $c = 120° 10' 3''$.

Ex. 675, $a = 89° 1'$, $b = 44° 0' 7''$, $c = 67°$.

Ex. 676, $a = 92° 6' 16''$, $b = 84° 0' 1''$, $c = 88° 7'$.

Ex. 677, $A = 81° 3'$, $B = 84° 7' 55''$, $C = 25° 10'$.

Ex. 678, $A = 88° 2'$, $B = 91° 35' 33''$, $C = 49° 11' 20''$.

Ex. 679, $A = 128° 10'$, $B = 155° 6'$, $C = 133° 22'$.

Ex. 680, $A = 159° 17' 3''$, $B = 160°$, $C = 141° 24' 42''$.

Ex. 681, $A = 93° 0' 9''$, $B = 62° 17'$, $C = 81° 17'$.

Ex. 682, $a = 49° 16' 10''$, $b = 59° 10'$, $C = 79° 11'$.

Ex. 683, $b = 81° 16' 4''$, $c = 22° 7'$, $A = 94° 6'$.

Ex. 684, $b = 124° 16'$, $c = 99° 9'$, $A = 100° 13'$.

Ex. 685, $a = 12° 19' 2''$, $c = 55° 7'$, $B = 44° 4' 2''$.

Ex. 686, $a = 88° 8'$, $c = 24° 19' 9''$, $B = 10° 6' 1''$.

Ex. 687, $A = 9° 59' 40''$, $B = 71° 19'$, $c = 63° 19'$.

Ex. 688, $B = 94° 11' 2''$, $C = 122° 15'$, $a = 44° 4' 2''$.

Ex. 689, $A = 13° 6' 21''$, $C = 87° 10'$, $b = 50° 50'$.

Ex. 690, $A = 77° 17' 22''$, $B = 4° 24'$, $c = 83° 10' 5''$.

Ex. 691, $A = 88° 6'$, $B = 139° 19'$, $c = 99° 16'$.

CHAPTER VII.

GEOMETRICAL PROGRESSION.

[In the examples in this chapter and the next, seven-place logarithms will be needed when accurate results are required. The computer, however, should not use the larger tables unnecessarily, as the work is greater with them.

In many computations four-place logarithms may be used, and even three-place tables are useful for rough preliminary calculations. Peirce's three and four-place tables are convenient for such work.]

The sum of n-terms in a geometrical progression of which the first term is a, and the ratio r, is $S = \dfrac{a(r^n - 1)}{r - 1}$, and the nth term is $l = ar^{n-1}$.

Of the five quantities a, r, n, l, and S, therefore, any three being given, the other two can be found. There thus occur ten cases, of which, as shown in many elementary works on algebra, the following twenty equations are the solutions.

1. $l = ar^{n-1}$; $S = \dfrac{ar^n - a}{r - 1}$ to find l and S.

2. $S = \dfrac{lr - a}{r - 1}$; $r^{n-1} = \dfrac{l}{a}$ " S " n.

3. $l=\dfrac{a+S(r-1)}{r}$; $r^n=\dfrac{a+S(r-1)}{a}$ to find l and n.

4. $r=\sqrt[n-1]{\dfrac{l}{a}}$, $S=\dfrac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}$, " r " S.

5. $ar^n-rS=a-S$; $l\,(S-l)^{n-1}=a\,(S-a)^{n-1}$ to find r and l.

6. $r=\dfrac{S-a}{S-l}$; $\left(\dfrac{S-a}{S-l}\right)^{n-1}=\dfrac{l}{a}$, " r " n.

7. $a=\dfrac{l}{r^{n-1}}$; $S=\dfrac{l(r^n-1)}{r^n-r^{n-1}}$, " a " S.

8. $a=\dfrac{S\,(r-1)}{r^n-1}$; $l=\dfrac{(r-1)\,S\,r^{n-1}}{r^n-1}$, " a " l.

9. $a=lr-S\,(r-1)$; $r^{n-1}=\dfrac{l}{lr-S\,(r-1)}$, to find a and n.

10. $a\,(S-a)^{n-1}=l(S-l)^{n-1}$; $(S-l)\,r^n-S\,r^{n-1}=l$, to find a and r.

Of these equations, all except 5 and 10 can be solved by means of logarithms. A few examples will illustrate the application of logarithms to such formulas.

Ex. Required the 10th term and the sum of 10 terms in the progression 3, 9, 27........

$l = 3 \times 3^9.$

$$
\begin{array}{rll}
& \text{Log } a = \log 3, & 0.4771213 \\
(n-1) & \text{Log } r = 9 \log 3, & \underline{4.2940917} \\
& \text{Log } l, & 4.7712130
\end{array}
$$

$$l = 59049.$$

Then,

$$S = \frac{ar^n - a}{r-1} \therefore \text{Log S} = \log (ar^n - a) - \log (r-1).$$

$$
\begin{array}{ll}
\text{Log } a = \log 3, & 0.4771213 \\
n\log r = 10 \log 3, & \underline{4.7712130} \\
& 5.2483343 \\
a\,r^n = 177{,}147 & \\
a\,r^n - a = 177{,}144 & \\
\log (ar^n - a) & 5.2483265 \\
\log (r-1) = \log 2 & \underline{0.3010300} \\
& 4.9472965
\end{array}
$$

$$S = 88572.$$

Ex. 692, $a = 2$, $r = 3$, $n = 10$, find l and S.

Ex. 693, $a = 24$, $r = \frac{1}{2}$, $n = 8$, " "

Ex. 694, $a = 3$, $r = 2$, $n = 9$, " "

Ex. Required the number of terms and the sum of the geometrical progression of which the first term is 5, the ratio 3, and the last term 98415.

$$r^{n-1} = \frac{l}{a}, \text{ hence } (n-1) \log r = \log l - \log a.$$

$$n - 1 = \frac{\log l - \log a}{\log r}.$$

$$\log (n-1) = \log [\log l - \log a] - \log \log r.$$

$$\begin{array}{rlcl}
\log l & 4.9930613 & & \\
\log a & \underline{0.6986700} & & \\
\log [\log l - \log a] = \log & 4.2940913 & = & 0.6328712 \\
\log \log r = \log & 0.4771213 & = & \underline{\bar{1}.6786288} \\
\log (n-1) & & & 0.9542424 \\
n - 1 = 9.0. & & & \\
n = 10. & & &
\end{array}$$

S is easily found from Eq. (2) without logarithms, and is 1407620.

Ex. 695, $a = 1, l = 4096, r = 2,$ to find n and S.

Ex. 696, $a = 1.25, l = 1280, r = 2,$ " "

Ex. 697, $a = 59049, l = 3, r = \frac{1}{3},$ " "

Ex. 698, $a = 7, r = 2, S = 889,$ to find l and n.

Ex. 699, $S = 292968, r = \frac{1}{5}, a = 234375,$ " "

Ex. 700, $a = 0.5, r = \frac{1}{2}, S = 0.546875,$ " "

Ex. Given, first term 1536, last term 0.75, and number of terms 12. Find the ratio and the sum.

$$r = \sqrt[n-1]{\frac{l}{a}} = \sqrt[11]{\frac{0.75}{1536}} \therefore$$

$$\log r = \tfrac{1}{11}[\log 0.75 - \log 1536],$$

from which we easily find, $r = 0.5 = \frac{1}{2}$. Then,

$$S = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}. \quad \text{Hence,}$$

$$\log S = \log\left[l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}\right] - \log\left[l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}\right].$$

$$\begin{array}{lrr}
\text{Log } 0.75\ldots\ldots & \bar{1}.8750513 & \\
 & 12 & \\
 & 11)\overline{\bar{2}.5007356} & \\
\text{Log } l^{\frac{n}{n-1}}\ldots\ldots & \bar{1}.8637032 & \\
l^{\frac{n}{n-1}}\ldots\ldots\ldots\ldots\ldots\ldots & & 0.730640 \\
\text{Log } 1536\ldots\ldots & 3.1863912 & \\
 & 12 & \\
 & 11)\overline{38.2366944} & \\
\text{Log } a^{\frac{n}{n-1}}\ldots\ldots & 3.4760631 & \\
a^{\frac{n}{n-1}}\ldots\ldots\ldots\ldots\ldots\ldots & & \underline{2992.700000} \\
l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}\ldots\ldots\ldots\ldots & & -2991.96936
\end{array}$$

$$\text{Log}\left[l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}\right] \ldots\ldots\ldots\ldots\ldots \; 3.4759572n$$

$$\begin{array}{l}
\text{Log } 0.75 \ldots\ldots \bar{1}.8750613 \\
\qquad\qquad\qquad\quad \tfrac{1}{11} \\
\qquad\qquad\qquad \overline{\bar{1}.9886419}
\end{array}$$

$$l^{\frac{1}{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 0.974185$$

$$\begin{array}{l}
\text{Log } 1536 \ldots\ldots 3.1863912 \\
\qquad\qquad\qquad\quad \tfrac{1}{11} \\
\qquad\qquad\qquad \overline{0.2896719}
\end{array}$$

$$a^{\frac{1}{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; \underline{1.948375}$$

$$l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}} \ldots\ldots\ldots\ldots \; -\,0.974190$$

$$\text{Log}\left[l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}\right] \ldots\ldots\ldots\ldots\ldots \; \underline{\bar{1}.9886437n}$$

$$3.4873135$$

$$S = 3071.25.$$

Ex. 701, $a = 4$, $l = 8748$, $n = 8$, to find r and S.
Ex. 702, $a = 0.6$, $l = 38.4$, $n = 7$, " "
Ex. 703, $a = 0.9$, $l = 0.0111111$, $n = 5$, " "

Ex. Given $a = 3$, $l = 1171875$, $S = 1464843$. Required r and n.

$r = \dfrac{S - a}{S - l}$; $\left(\dfrac{S - a}{S - l}\right)^{n-1} = \dfrac{l}{a}$. From the first we easily find $r = 5$. Then, from the second,

$$(n-1)\,[\log(S-a)-\log(S-l)]=\log l-\log a,$$

whence, $\quad \log(n-1)=\log\,[\log l-\log a]$

$$-\log\,[\log(S-a)-\log(S-l)].$$

Log 1171875 . . . . 6.0688813
Log 3 . . . . 0.4771213

Log [log l − log a] = log 5.5917600 = 0.7475485
Log 1464840 . . . . 6.1657902
Log 292968 . . . . 5.4668202

Log [log (S − a) − log (S − l)]
= log 0.6989700 = $\overline{1}$.8444585
log (n − 1) 0.9030900
$n - 1 = 8.$
$n = 9.$

Ex. 704, $a = 81$, $l = 0.111111$, $S = 121.444444$, find n and r.

Ex. 705, $a = 0.7$, $l = 13778.1$, $S = 20666.8$, find n and r.

Ex. 706, $a = 1.25$, $l = 3906.25$, $S = 4882.5$, find n and r.

Ex. 707, $l = 0.44375$, $n = 5$, $r = \frac{1}{2}$, find a and S.

Ex. 708, $l = 91.8784$, $n = 7$, $r = 2$, " "

Ex. 709, $l = 0.196349$, $n = 5$, $r = \frac{1}{2}$, " "

Ex. 710, $n = 6$, $r = \frac{1}{2}$, $S = 11.8125$, find a and l.

Ex. 711, $n = 5$, $r = 3$, $S = 10.4567787$, " "

Ex. 712, $n = 6$, $r = 2$, $S = 19.53125$, " "

Ex. Given $l = 49152$, $r = 4$, $S = 65535$.

We find, from $a = lr - S\ (r-1)$, $a = 3$.

From $r^{n-1} = \dfrac{l}{lr - S\ (r-1)} = \dfrac{l}{a}$, we have,

$$\log\ (n-1) = \log\ [\log l - \log a] - \log \log r,$$

the solution of which gives,

$$n - 1 = 7$$
$$n = 8.$$

Ex. 713, $l = 7.8125$, $r = \frac{1}{2}$, $S = 242.1875$, find a and n.
Ex. 714, $l = 300$, $r = 2$, $S = 590.625$, " "
Ex. 715, $l = 117649$, $r = 7$, $S = 137257$, " "

CHAPTER VIII.

INTEREST AND ANNUITIES.

COMPOUND INTEREST.

THE amount a, of a principal p, at compound interest for t years at r per cent, is $a = p\,(1 + r)^t$.

From this, either a, p, r or t may be found, when the others are known, by the equations,

(1.) $\log a = \log p + t \log\,(1 + r)$.

(2.) $\log p = \log a - t \log\,(1 + r)$.

(3.) $\log\,(1 + r) = \dfrac{\log a - \log p}{t}$.

(4.) $t = \dfrac{\log a - \log p}{\log\,(1 + r)}$; or,

$$\log t = \log\,[\log a - \log p] - \log \log\,(1 + r).$$

Ex. What will be the amount of \$175.25 at compound interest for 15 years, at 4 per cent?

Formula (1.)

Log p		2.2436580
Log $(1 + r)$	0.0170333	
15 log $(1 + r)$		0.2554995
	Log a	2.4991575

$a = \$315.61\frac{1}{2}$.

The remark made on involution, page 45, applies here. A table of ten-place logarithms of ordinary values of $(1 + r)$, is given in "Loomis' Tables," page 150, but it will not be needed, except for very long periods of years. The result obtained in the last problem by this table would differ from that found, by less than half a cent.

Ex. What capital must be invested, at 4 per cent, to amount to $25,000 in 12 years?

Formula (2.)

Log 25000		4.3979400
Log 1.04	0.017033	
	12	
		0.2043996
		4.1935004

$p = \$1561.31$.

Ex. At what rate must $10,000 be invested, to amount, in 100 years, to $100,000?

$$\begin{array}{lr}
\text{Log } 100000 & 5.0000000 \\
\text{Log } 10000 & 4.0000000 \\
\hline
& 100)\overline{1.0000000} \\
\text{Log } (1+r) = & 0.010000
\end{array}$$

$$1 + r = 1.0233,$$

$$r = 0.0233 \text{ or } 2\tfrac{33}{100} \text{ per cent.}$$

Ex. In what time will a capital double itself at $3\frac{1}{2}$ per cent?

$$t = \frac{\log 2p - \log p}{\log(1+r)} = \frac{\log 2 + \log p - \log p}{\log(1+r)} = \frac{\log 2}{\log(1+r)}$$

$$\log t = \log \log 2 - \log \log (1+r).$$

$$\begin{array}{lr}
\text{Log log } 2 & \bar{1}.4786098 \\
\text{Log log } 1.035 & \bar{2}.1743593 \\
\hline
& 1.3042505
\end{array}$$

$$t = 20.15 \text{ years.}$$

The same formulas will apply when the interest is paid semi-annually, provided t represent the number of half years, and r half the rate per cent.

Ex. 716. What will be the amount of \$156 at compound interest for 40 years at 5 per cent?

Ex. 717. At what rate must interest be paid on \$3,550, in order that it may amount in 15 years to \$8,000?

Ex. 718. What capital must be invested at 4 per cent, compounded semi-annually, to amount in 20 years to \$16,000?

Ex. 719. In what time will $1,500 at 4 per cent, compounded semi-annually, amount to $6,000?

Ex. 720. In what time will a capital at 5 per cent, compounded annually, multiply itself one hundred-fold?

Ex. 721. At what rate must a capital be invested, to quadruple itself once in every century?

Ex. 722. What capital must be invested at 7 per cent, to amount to $5,000,000 in 200 years?

ANNUITIES.

The amount of an annuity for t years, is shown in treatises on algebra to be $A = \frac{a\,[(1+r)^t - 1]}{r} \ldots$ (A.)

and the present worth, $p = \frac{a}{r}\left[\frac{(1+r)^t - 1}{(1+r)^t}\right] \ldots$ (B.)

in which a is the annual payment or annuity, and r the rate of interest. Any two of the five quantities can be found if the other three are given. There thus arise ten cases, namely,

1.	Given	a, r, p,	to find	A	and	t,
2.	"	a, r, A,	"	p	"	t,
3.	"	a, r, t,	"	A	"	p,
4.	"	a, p, t,	"	A	"	r,
5.	"	A, r, p,	"	a	"	t,
6.	"	A, r, t,	"	a	"	p,
7.	"	r, p, t,	"	a	"	A,
8.	"	p, a, A,	"	t	"	r,
9.	"	p, a, t,	"	A	"	r,
10.	"	t, a, A,	"	p	"	r,

Of these, 9 and 10 can only be solved approximately.* Of the others, the following solutions are easily found:

1. Dividing (A) by (B), we find $\frac{A}{p} = (1 + r)^t$; and from (A)

$$\frac{A\,r}{a} + 1 = (1 + r)^t, \text{ whence } A = \frac{a\,p}{a - r\,p} \quad \ldots\ldots\ldots (1)$$

Also, $\frac{A}{p} = (1 + r)^t = \frac{a}{a - r\,p}$, whence

$$t = \frac{\log a - \log (a - rp)}{\log (1 + r)} \quad \ldots\ldots\ldots (2)$$

By similar processes we find,

$$2.\ t = \frac{\log (A\,r + a) - \log a}{\log (1 + r)} \quad \ldots\ldots\ldots (3)$$

$$p = \frac{A\,a}{A\,r + a} \quad \ldots\ldots\ldots (4)$$

$$3.\ A = \frac{a\,[(1 + r)^t - 1]}{r} \quad \ldots\ldots\ldots (5)$$

$$p = \frac{a\,[(1 + r)^t - 1]}{r\,(1 + r)^t} \quad \ldots\ldots\ldots (6)$$

$$4.\ \log (1 + r) = \frac{\log A - \log p}{t} \quad \ldots\ldots\ldots (7)$$

$$a = \frac{A\,r}{(1 + r)^t - 1} \quad \ldots\ldots\ldots (8)$$

---

* See Callet's Logarithms, xxi.

5. $$t = \frac{\log A - \log p}{\log (1 + r)} \qquad (9)$$

$$a = \frac{A r p}{A - p} \qquad (10)$$

6. $$a = \frac{A r}{(1 + r)^t - 1} \qquad (11)$$

$$p = \frac{A}{(1 + r)^t} \qquad (12)$$

7. $$A = p\,(1 + r)^t. \qquad (13)$$

$$a = \frac{p r\,(1 + r)^t}{(1 + r)^t - 1} \qquad (14)$$

8. $$r = \frac{a\,(A - p)}{A p} \qquad (15)$$

$$t = \frac{\log A - \log p}{\log (1 + r)} \qquad (16)$$

(The solution of (8) requires the previous solution of (7).)

The methods of solution of the above equations by means of logarithms are obvious. A few examples will suffice to illustrate the arrangement of the computation.

Ex. Required the amount due on an annuity of \$1,500 which has been unpaid for 12 years, interest being computed at 4 per cent.

$\text{Log}\,(1 + r)^t = t \log (1 + r) = 12 \log 1.04 = 0.2043996.$

Hence $(1 + r)^t = 1.601$, and $(1 + r)^t - 1 = 0.601$.

Then, Eq. (5).

Log 1500	3.1760913
Log 0.601	$\bar{1}$.7788745
Colog 0.04	11.3979401
Log A	4.3529059

A = \$22,537.50.

Ex. Required the present worth of an annuity of \$225, to continue for 12 years, interest being computed at 4 per cent.

Eq. (6). $\text{Log}\, p = \log 225 + \log [(1.04)^{12} - 1]$
$- \log 0.04 - \log [(1.04)^{12}]$.

Log 1.04	0.0170333
	12
Log 1.04^{12}	0.2043996

$1.04^{12} = 1.601 \therefore 1.04^{12} - 1 = 0.601$. Then,

Log 225	2.3521825
Log $[(1.04)^{12} - 1]$	$\bar{1}$.7788745
Colog 0.04	11.3979401
Colog 1.601	9.7956087
	3.3246058

$p =$ \$2,111.57.

Ex. What annuity can be purchased for \$10,000, to run 20 years, money being worth 4 per cent?

$$20 \times \log 1.04 \qquad 0.3406667$$
$$1.04^{20} = 2.191$$
$$(1+r)^t - 1 = 1.191$$

Then, Eq. (14).

Log 10000	4.0000000
Log 0.04	$\bar{2}.6020599$
20 log 1.04	0.3406667
Colog 1.191	9.9240882
	2.8668148

$$a = \$735.89$$

Ex. 723. How much must be given for an annuity of $1,000, to continue 21 years, interest being reckoned at 4 per cent?

Ex. 724. What annuity can be bought for $10,000, to run 15 years, interest being computed at 5 per cent?

Ex. 725. At what rate must interest be computed, if an annuity of $617.87, to continue 21 years, is to be purchased for $10,000?

Ex. 726. For what time will an annuity of $500 run, the price paid for it being $8,000, and interest being reckoned at 5 per cent?

Ex. 727. If I pay $16,890.35 for an annuity of $1,000 to run 26 years, at what rate is interest reckoned?

Ex. 728. To how much will an annuity of $1,120 amount in 12 years, interest being computed at 4 per cent?

Ex. 729. What annuity can be purchased for $20,000, to run 20 years, interest being reckoned at $3\frac{1}{2}$ per cent?

For other problems in interest and annuities, and for tables for solving such problems, see Jones, *On the Value of Annuities.*

ANSWERS TO PROBLEMS.

Ex. 22, 3.420415.
23, 3.890086.
24, 3.950073.
25, 3.271609.
26, 3.330008.
37, 5.700617.
38, 4.760347.
39, 4.750045.
40, 3.659916.
41, 5.990072.
52, 0.729246.
53, 0.907465.
54, 0.954677.
55, 0.786822.
56, 1.970021.
67, $\bar{1}$.158362.
68, $\bar{2}$.173186.
69, $\bar{5}$.903633.
70, $\bar{3}$.907465.
71, $\bar{5}$.999565.
82, 5.959519.
83, 6.845470.
84, 6.830075.
85, 6.890132.
86, 6.860098.
87, 7.310058.
88, 7.690026.
89, 5.910058.

Ex. 90, 6.669203.
91, 4.830069.
92, 6.463759.
93, 5.000816.
94, 4.267383.
95, 5.284207.
96, 5.917652.
97, 0.497150.
98, 2.562582.
99, 1.592610.
100, 1.507597.
101, 0.515993.
102, 1.595174.
103, $\bar{1}$.793371.
104, $\bar{1}$.484000.
105, 0.404817.
106, 0.206637.
107, 0.392873.
108, 2.077731.
109, $\bar{1}$.657343.
110, 1.547978.
111, $\bar{1}$.958086.
112, 0.023952.
113, $\bar{1}$.944680.
114, $\bar{1}$.883377.
115, 0.042063.
116, 0.055340.
117, 0.062356.

Ex. 118, 1.547036.
119, 1.560301.
120, 0.559238.
121, 1.188430.
122, $\bar{2}$.547775.
123, 0.343331.
124, $\bar{1}$.656663.
125, $\bar{1}$.571988.
126, $\bar{1}$.285445.
127, 0.714547.
148, 542626.3.
149, 410185.8.
150, 17025.0.
151, 13557.5.
152, 302796.3..
153, 32361.5.
154, 426.6.
155, 44.674.
156, 4.575.
157, 323.674.
158, 0.037170.
159, 0.105052.
160, 0.100241.
161, 0.100031.
162, 0.080938.
163, 0.794474.
164, 0.005630.
165, 0.049081.
166, 0 310671.
167, 0.316681.
168, 0.010729.
169, 768417; true ans. 768416.
170, 970162; " 970160.
171, 954806; " 954805.
172, 1488695; " 1488694.
173, 586179; " 586180.

Ex. 174, 5581400; tr. an. 5581404.
175, 5540110; " 5540112.
176, 4441171; " 4441184.
177, 319328.
178, 194102.
179, 14259.56.
180, 60250.70.
181, 26017.56.
182, 41.12.
183, 1.0676.
184, 51.5882.
185, 3001034.
3001000.*
3001000.03.**
186, 382727.9.
382728.3.* **
187, 5858540.
5858515.* **
188, 20.79670.
20.79660.* **
189, 8.1359.
190, 44.435.
191, 0.3102.
192, 0.00004569.
193, 0.00204.
194, 2.568.
195, 0.5725.
196, 0.000643.
197, 0.001243.
198, 0.7183.
199, -4.75.
200, 0.2332.
201, -0.03717.
202, -0.0002967.
203, -1.552.
204, 220.44.

* By seven-place logarithms. ** True answer.

Ex. 205,	6.507.	Ex. 240,	0.06062.
206,	0.1155.	241,	0.014402.
207,	− 0.000858.	242,	0.05772.
208,	− 0.03164.	243,	108.368.
209,	− 1024754.	244,	0.02840.
	− 1024758.*	245,	1.028.
210,	0.001097.	246,	1.4990.
211,	− 1.0963.	247,	1086.55.
212,	0.001791.	248,	5.010.
213,	44.773.	249,	1.2308.
214,	796.91.	250,	0.0000010825.
215,	15.247.	251,	63.088.
216,	2249.4.	252,	0.09240.
217,	10732.8.	253,	0.8525.
218,	15.717.	254,	0.00450.
219,	115.596.	255,	0.00005657.
220,	22.078.	256,	0.06124.
221,	1.156.	257,	0.0007872.
222,	10.45.	258,	22.713.
223,	6.0805.	259,	0.4951.
224,	5.0136.	260,	0.4289.
225,	9.2556.	261,	0.5025.
226,	87.778.	262,	0.9337.
227,	5.9531.	263,	119.96.
228,	7.2425.	264,	0.2693.
229,	8.9073.	265,	0.01123.
230,	65.911.	266,	1.0942.
231,	797.970.	267,	80055.
232,	1.2378.	268,	88.88.
233,	0.008883.	269,	0.5820.
234,	0.06514.	270,	0.02031.
235,	0.08417.	271,	1.3499.
236,	0 08241.	272,	940.334.
237,	0.09895.	273,	− 17993.4.
238,	0.09131.	274,	− 0.6276.
239,	0.08864.	275,	94431.0.

* By seven-place logarithms.

Ex. 276,	409.1.	Ex. 312,	809.02.
277,	2912334000000.*	313,	50096800000.
278,	10347173.*	314,	2.97753.
279,	10544300000000.*	315,	1.66665.
280,	19770600.*	316,	194.519.
281,	1.015.	317,	48927.2.
282,	87.903.	318,	− 1.0242.
283,	3207197.*	319,	65966892.**
284,	90017200.*	320,	− 351.76.
285,	24.626.	321,	− 1345.57.
286,	3449903000.*	322,	4610074000.
287,	0.0003521.	323,	1219.97.
288,	0.002495.	324,	− 1.5457.
289,	0.91743.	325,	478.78.
290,	0.06172.	326,	288744.8.
291,	0.0054803.	327,	− 0.0000000002003.
292,	0.72431.	328,	0.000005994.
293,	0.00059168.	329,	− 0.0008451.
294,	0.909099.	330,	0.01046.
295,	0.00012462.	331,	− 0.000009116.
296,	0.0001891.	332,	0.1098.
297,	0.0000001250.	333,	0.000009094.
298,	0.83905.	334,	− 0.007701.
299,	0.002430.	335,	− 0.00000006302.
300,	0.5955.	336,	− 0.000002137.
301,	0.00001422.	337,	1.0014.
302,	0.001742.	338,	24029219.
303,	0.0000002016.	339,	− 0.005929.
304,	0.00006435.	340,	− 0.06266.
305,	0.5205.	341,	− 1.2884.
306,	0.06395.	342,	− 0.00019422.
307,	372352.6.	343,	0.0100091.
308,	1.99472.	344,	− 13.4015.
309,	190.414.	345,	1.659.
310,	1047.33.	346,	95.3674.
311,	1.0713.	347,	1.31815.

* Correct to seven figures. ** By seven-place logarithms.

Ex. 348, 2.5181.
349, 1.8126.
350, 2.5414.
351, 4.9986.
352, 16.183.
353, 7.147.
354, 3.8023.
355, 5.2785.
356, 1.2023.
357, 1.2092.
358, 0.97245.
359, 0.40771.
360, 0.30731.
361, 0.37097.
362, 0.10723.
363, 0.44235.
364, 0.01496.
365, 0.11122.
366, 0.76290.
367, 0.23620.
368, 0.1257.
369, 0.23401.
370, 0.98253.
371, 0.074532.
372, 0.3918.1
373, 0.92309.
374, 0.21888.
375, 0.21451.
376, 0.68394.
377, 0.34352.
378, 2.6152.
379, 1.1272.
380, 1.8009.
381, 1.0381.
382, 1.3688.
383, 5.6558.

Ex. 384, 2.5962.
385, 2.9863.
386, 2.0573.
387, 1.2239.
388, − 0.42197.
389, − 0.94671.
390, Imaginary.
391, − 2.84275.
392, − 0.65765.
393, − 0.42461.
394, − 1.0881.
395, Imaginary.
396, Imaginary.
397, − 1.3583.
398, 11464848.
399, 3.2769.
400, 5.4093.
401, 4.3014.
402, 12361.9.
403, 1.3791.
404, 1.00935.
405, 1.5919.
406, 3.8885.
407, 686.125.
408, 2466281200.*
409, 6858.93.*
410, 395.478.*
411, 0.00066422.*
412, 36134603*...to 23 figs.
413, 273004090.*
414, 4940.37.*
415, 19975.77.*
416, 0.61873.*
417, 6245.84.*
418, 88016...to 11 figures.
419, 30771.7.

* By seven-place logarithms.

Ex. 420, 15734...to 9 figures.
421, 1.9641.
422, 2111300.
423, 45768...to 101 figures.
424, 5.6255.
425, 2465.58.
426, 4.642.
427, 1.3082.
428, 0.87193.
429, 0.0000005119.
430, 0.019741.
431, 0.0027273.
432, 0.20177.
433, 0.9366.
434, 0.005866.
435, 0.024337.
436, 0.4357.
437, 0.4116.
438, 0.95732.
439, 0.27023.
440, 0.10312.
441, 0.995290.
442, 0.9954689.
443, 0.9956322.
444, 0.9997890.
445, 0.9970800.
446, 0.834336.
447, 0.011101.
448, 0.010173.
449, 0.035063.
450, 0.011087.
451, 0.58773.
452, 0.056314.
453, 0.0012812.
454, 0.20685.
455, 0.4709.
456, 0.11967.
457, 0.86441.

Ex. 458, 0.044478.
459, 0.00031381.
460, 0.98430.
461, 0.29858.
462, 0.000020858.
463, 0.0030140.
464, 0.0045695.
465, 0.000029004.
466, 0.00000026106.
467, 0.072987.
468, 0.010793.
469, 0.95247.
470, 0.0030851.
471, 0.894375.
472, 0.22897.
473, 2.94970.
474, 0.0029992.
475, 2.06183.
476, 5.13642.
477, 1.48516.
478, 5.8280.
479, 8.8609.
480, 3.2801.
481, 1.4398.
482, 1.2097.
483, 1.3322.
484, 1.5017.
485, 2.1194.
486, 1.3713.
487, 1.5249.
488, 0.92549.
489, 0.66738.
490, 0.2090.
491, 0.6819.
492, 0.6519.
493, 0.6503.
494, 0.1619.
495, 0.8683.

Ex. 496, 0.6499.
497, 0.9526.
498, 1.8565.
499, 1.2535.
500, 1.4346.
501, 1.6850.
502, 0.39768.
503, 1.4409.
504, 3.1889.
505, 2.
506, 3.4307.
507, 3.0460.
508, 0.41141.
509, 0.19770.
510, 0.047697.
511, 0.37478.
512, 0.82282.
513, 0.081910.
514, 0.087510.
515, 0.35655.
516, 0.35470.
517, 0.15200.
518, − 0.043921.
519, − 0.35811.
520, − 0.75237.
521, − 0.43297.
522, − 0.69393.
523, − 1.73522.
524, − 0.15336.
525, − 1.96149.
526, − 1.36882.
527, − 0.88520.
528, 0.284764.
529, 0.677710.
530, 2062.65.
531, 1.07975.
532, 0.999659.
533, 0.941436.

Ex. 534, 0.999984.
535, 0.207920.
536, 9.93391.
537, 0.002836.
538, 0.710205.
539, 0.859876.
540, 1.19033.
541, 0.999997.
542, 0.981075.
543, 0.258451.
544, 0.187878.
545, 0.003103.
546, 0.031798.
547, 0.549807.
548, 18° 38′ 52″.4.
549, 56° 38′ 53″.9.
550, 86° 30′ 30″.
551, 89° 10′ 41″.
552, 89° 57′ 56″.5.
553, 0° 2′ 29″.
554, > 89° 42′ 15″.
< 89° 42′ 45″.
555, > 0° 9′ 30″.
< 0° 10′ 30″.
556, 69° 47′ 0″.9.
557, 44° 33′ 50″.6.
558, 30° 40′ 3″.9.
559, 68° 46′ 35″.2.
560, 27° 48′ 15″.2.
561, 44° 30′ 47″.5.
562, 45° 1′ 1″.7.
563, 88° 13′ 0″.03.
564, 3° 20′ 10″.6.
565, 0° 11′ 53″.5.
566, 84° 53′ 40″.3.
567, 3° 48′ 34″.5.
568, 0.939255.
569, − 0.978935.

Ex. 570, − 19.2959.
571, − 2.82000.
572, 0.189155.
573, − 0.994741.
574, − 0.519022.
575, − 0.872864.
576, 0.789087.
577, − 11.2417.
578, 44° 33′ 24″.7; or, 315° 26′ 35″.3.
579, 87° 20′ 9″.8; or, 217° 20′ 9″.8.
580, 209° 58′ 29″.9; or, 330° 1′ 30″.1.
581, 152° 2′ 15″.4; or, 207° 49′ 44″.6.
582, 94° 52′ 12″.3; or, 274° 52′ 12″.3.
583, 158° 0′ 17″.1; or, 238° 0′ 17″.1.
584, 151° 58′ 12″.0; or, 208° 1′ 48″.0.
585, 58° 53′ 58″.8; or, 301° 6′ 1″.2.
586, 19° 20′ 58″.1; or, 160° 39′ 1″.9.
587, 204° 18′ 43″.1; or, 335° 41′ 16″.9.
588, $b =$ 317.8.
589, $b =$ 85.59, $c =$ 95.08.
590, $a =$ 15483.73, $c =$ 15572.5.
591, $a =$ 333.59, $c =$ 356.7.
592, $a =$ 1415.55, $b =$ 124.33.
593, A = 12° 50′ 49″.3. B = 77° 9′ 10″.7.
594, A = 71° 18′ 41″.5. B = 18° 41′ 18″.5.
595, A = 67° 30′ 46″.8. B = 22° 29′ 13″.2.
596, $a =$ 190.19. $c =$ 274.78.
Ex. 597, $a =$ 6.895. $b =$ 18.343.
598, B = 28° 19′ 21″.1. C = 131° 23′ 24″.9. $c =$ 116.92.
599, A = 37° 38′ 16″.5. B = 113° 2′ 18″.5. $c =$ 115.5.
600, 8.721672.
601, 9.599686.
602, 9.570614.
603, 8.955885.
604, 9.076432.
605, 9.999975.
606, 11.995523.
607, 9.999896.
608, 11.516192.
609, 11.027145.
610, 24° 30′ 25″.2.
611, 33° 27′ 10″.9.
612, 31° 39′ 6″.0.
613, 39° 45′ 8″.0.
614, 0° 53′ 50″.0.
615, 43° 34′ 31″.0.
616, 34° 40′ 1″.2.
617, 1° 6′ 6″.5.
618, > 0° 20′ 10″. < 0° 20′ 40″.
619, > 7° 20′ 30″. < 7° 20′ 36″.
620, 9.713935.
621, 10.153161n
622, 10.077726.
623, 9.914422n.
624, 9.530448n.
625, 9.931840.
626, 9.625265.
627, 9.997570n.
628, 10.836877.
629, 9.986394n.

Ex. 630, $92° \ 6' \ 59''.7$; or, $272° \ 6' \ 59''.7$.
631, $7° \ 38' \ 2''.4$; or, $172° \ 21' \ 57''.6$.
632, $73° \ 11' \ 33''.3$; or, $286° \ 48' \ 26''.7$.
633, $161° \ 20' \ 56''.5$; or, $341° \ 20' \ 56''.5$.
634, $91° \ 9' \ 45''.5$; or, $188° \ 50' \ 14''.5$.
635, $178° \ 58' \ 51''.2$; or, $358° \ 58' \ 51''.2$.
636, $181° \ 29' \ 50''.7$; or, $358° \ 30' \ 9''.3$.
637, $169° \ 8' \ 11''.0$; or, $349° \ 8' \ 11''$.
638, $22° \ 11' \ 20''.3$; or, $202° \ 11' \ 20''.3$.
639, $103° \ 55' \ 39''.5$; or, $256° \ 4' \ 20''.5$.
640, $A = 57° \ 29' \ 8''$, $B = 32° \ 30' \ 51''$, $a = 33.83$.
641, $b = 698.3$, $a = 569.6$, $B = 50° \ 47' \ 41''$.
642, $A = 27° \ 50' \ 38''$, $B = 62° \ 9' \ 22''$, $b = 55.4$.
643, $A = 1° \ 49' \ 32''$, $a = 51.02$, $b = 1600.2$.
644, $A = 2° \ 35' \ 49''.5$, $B = 87° \ 24' \ 10''.5$, $c = 450.4$.
645, $A = 41° \ 57' \ 10''.5$, $B = 48° \ 2' \ 49''.5$, $c = 74.49$.
646, $C = 108° \ 34' \ 36''$, $a = 67.02$, $b = 16.59$.
647, $C = 28° \ 47' \ 34''$, $a = 199.5$, $b = 116.9$.
648, $C = 20° \ 43' \ 23''$, $b = 372.0$, $c = 146.4$.
649, $c = 348.9$, $B = 3° \ 9' \ 44''.5$, $C = 152° \ 39' \ 0''.5$.
650, $c = 1690.4$, $B = 2° \ 46' \ 12''$, $C = 66° \ 11' \ 7''$.
651, $c = 34.53$, $A = 44° \ 40' \ 8''$, $B = 97° \ 7 \ 39''$.
652, $c = 564.8$, $A = 33° \ 27' \ 17''.5$, $B = 48° \ 28' \ 27''.5$.
653, $A = 32° \ 56' \ 21''.5$, $B = 28° \ 23' \ 5''.1$,
$C = 118° \ 39' \ 47''.5$.
654, $A = 89° \ 50' \ 47''$, $B = 1° \ 47' \ 10''$,
$C = 88° \ 22' \ 3''$.
655, $A = 15° \ 37' \ 0''$, $B = 22° \ 12' \ 49''$,
$C = 142° \ 10' \ 11''$.
656, $a = 39° \ 57' \ 54''$, $b = 144° \ 27' \ 52''$,
$B = 139° \ 33' \ 39''$.
657, $A = 60° \ 14' \ 19''$, $B = 146° \ 21' \ 43''$,
$b = 163° \ 32' \ 50''$.
658, $a = 39° \ 42' \ 32''$, $c = 41° \ 37' \ 23''$,
$A = 74° \ 7' \ 11''$.
659, $c = 66° \ 32' \ 0''.5$, $b = 134° \ 41' \ 50''$,
$A = 113° \ 57' \ 7''.5$.

Ex. 660, $A = 48° 33' 23''.5$, $B = 72° 19' 5''.3$,
$c = 73° 39' 59''$.

661, $a = 54° 24' 31''$, $b = 82° 13' 6''$,
$c = 60° 30'$.

662, $B = 35° 22' 7''.6$, $C = 102° 29' 7''.0$,
$c = 55° 33' 53''$.

663, $A = 164° 30' 1''$, $B = 104° 7' 27''$,
$b = 101° 48' 9''.2$;

or, $A = 15° 29' 59''$, $B = 2° 34' 23''$,
$b = 8° 16' 0''$.

664, $c = 30° 44' 45''$, $A = 124° 55' 17''$,
$C = 25° 19' 8''$.

665, $a = 50° 10' 50''$, $A = 26° 46' 13''.2$,
$C = 21° 43' 16''.6$.

666, $a = 96° 12' 0''$, $A = 63° 54' 42''$,
$B = 174° 35' 11''$.

667, $C = 97° 27' 23''.0$, $a = 21° 58' 47''.0$,
$c = 71° 43' 9''.2$.

668, $C = 22° 44' 39''$, $b = 60° 37' 16''$,
$c = 36° 29' 24''$.

669, Impossible.

670, $A = 55° 13' 33''.6$, $a = 40° 40' 4''.6$,
$b = 52° 7' 4''.0$;

or, $A = 140° 14' 47''.0$, $a = 149° 30' 43''.0$,
$b = 127° 52' 56''.0$.

671, $A = 78° 51' 34''$, $a = 79° 13' 24''$,
$c = 64° 15' 44''$.

672, $A = 17° 44' 38''$, $B = 10° 4' 42''.6$,
$C = 156° 36' 38''.0$.

673, $A = 94° 16' 45''$, $B = 100° 22' 45''$,
$C = 140° 10' 20''$.

674, $A = 132° 59' 16''$, $B = 139° 51' 52''$,
$C = 139° 18' 30''$.

675, $A = 114° 22' 26''$, $B = 39° 15' 41''.2$,
$C = 56° 59' 28''$.

676, $A = 92° 18' 55''$, $B = 83° 55' 24''$,
$C = 87° 53' 1''$.

Ex. 677,	$a = 54° \ 5' \ 25''$,	$b = 54° \ 38' \ 58''$,
		$c = 20° \ 24' \ 22''.6$.
678,	$a = 88° \ 47' \ 26''$,	$b = 90° \ 25' \ 16''$,
		$c = 55° \ 42' \ 15''$.
679,	$a = 89° \ 7' \ 15''$,	$b = 147° \ 37' \ 30''$,
		$c = 112° \ 27' \ 20''$.
680,	$a = 160° \ 17' \ 40''$,	$b = 160° \ 58' \ 26''$,
		$c = 36° \ 28' \ 12''$.
681,	$a = 88° \ 48' \ 51''$,	$b = 62° \ 24' \ 34''$,
		$c = 81° \ 57' \ 21''$.
682,	$c = 108° \ 52' \ 7''.6$,	$A = 56° \ 46' \ 57''.5$,
		$B = 71° \ 25' \ 69''.5$.
683,	$a = 83° \ 27' \ 6''$,	$B = 82° \ 54' \ 35''.2$,
		$C = 22° \ 12' \ 34''.2$.
684,	$a = 93° \ 9' \ 44''$,	$B = 125° \ 27' \ 21''$,
		$C = 103° \ 19' \ 7''$.
685,	$b = 46° \ 48' \ 16''$,	$A = 11° \ 44' \ 24''.9$,
		$C = 128° \ 30' \ 15''.3$.
686,	$b = 64° \ 9' \ 10''.8$,	$A = 169° \ 45' \ 19''.2$,
		$C = 4° \ 12' \ 9''.6$.
687,	$C = 112° \ 30' \ 37''.2$,	$a = 34° \ 34' \ 13''.0$,
		$b = 43° \ 22' \ 52''.0$.
688,	$A = 55° \ 26' \ 44''.0$,	$b = 122° \ 37' \ 33''.5$,
		$c = 134° \ 25' \ 31''.5$.
689,	$B = 84° \ 33' \ 24''$,	$a = 10° \ 10' \ 14''.5$,
		$c = 51° \ 3' \ 55''.1$.
690,	$C = 102° \ 9' \ 25''$,	$a = 82° \ 11' \ 56''.2$,
		$b = 4° \ 27' \ 44''.8$.
691,	$C = 94° \ 34' \ 31''$,	$a = 81° \ 42' \ 50''.1$,
		$b = 139° \ 48' \ 7''.7$.

692, 59050 ; 59048.*
693, $l = 0.1875$; $S = 47.8125$.
694, $l = 768.0$; $S = 1533$.
695, $n = 12$; $S = 8191$.
696, $n = 11$; $S = 2558.75$.
697, $n = 10$; $S = 88572$.

* True answer.

Ex. 698, $l = 448$;	$n =$	7.
699, $l = 8$;	$n =$	8.
700, $l = 0.003125$;	$n =$	5.
701, $r = 4$;	$S =$	13120.
702, $r = 2$;	$S =$	76.2.
703, $r = \frac{1}{3}$;	$S =$	1.34444.
704, $n = 7$;	$r =$	$\frac{1}{2}$.
705, $n = 10$;	$r =$	3.
706, $n = 6$;	$r =$	5.
707, $a = 7.1$;	$S =$	13.75625.
708, $a = 1.4356$;	$S =$	182.3212.
709, $a = 3.14159$;	$S =$	6.08683.
710, $a = 6$;	$l =$	0.1875.
711, $a = 0.0864197$;	$l =$	7.
712, $a = 0.46875$;	$l =$	15.
713, $a = 125$;	$n =$	5.
714, $a = 9.375$;	$n =$	6.
715, $a = 1$;	$n =$	7.

716, $1098.24.
717, $5\frac{1}{2}$ per cent.
718, $7246.25.
719, 35 years.
720, 94.4 years.
721, 1.4 per cent.
722, $6.64.
723, $14029.16.
724, $963.42.
725, $2\frac{1}{2}$ per cent.
726, 33 years.
727, 4 per cent.
728, $16828.90.
729, $1407.22.